A THEOREM ABOUT FRACTIONAL INTEGRALS

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It is a classical result that if $f(x)$ is Lebesgue integrable in a finite interval, then it is finite p.p. One is led to enquire about the behaviour of the fractional integral f_β of f.

Suppose, for convenience, that $f(x)$ is defined in $[0, 2\pi]$. We then have the following theorem.

THEOREM. *If* $f \in L^q[0, 2\pi]$, *then*:

(a) For $0<\alpha<1$, $2<q<\infty$, $f_{\alpha/q}$ *is finite everywhere except in a set which is of zero* β*-capacity for every* $\beta>1-\alpha$.

(b) *For* $0<\alpha<1$, $1\leqq q\leqq 2$, $f_{\alpha/q}$ *is finite everywhere except possibly in a set of zero* $(1-\alpha)$*-capacity.*

Both (a) *and* (b) *are best possible.*

Since, as is well known, the Riemann-Liouville and the Weyl versions of the fractional integral of a function in L^q differ by a bounded function, this theorem holds for both versions if it is shown to hold for either one. Use is made of this fact in what follows.

1. In this section I prove a lemma which is possibly of greater interest than the theorem itself.

LEMMA. *Let* $\mu(x)$ *be a nondecreasing bounded function in* $[0, 2\pi]$. *Let, for* $0<\alpha<1$,

$$V_{1-\alpha} = \sup \int_0^{2\pi} | x - t |^{\alpha-1} d\mu(t) \quad \textit{for } x \in [0, 2\pi].$$

Then, for every $\epsilon>0$ *and* $1<q<2$,

$$M_{q-\epsilon}\left[\int_0^{2\pi} | x - t |^{\alpha/q'-1} d\mu(t) \right] \leqq A(\alpha, \epsilon) V_{1-\alpha}^{1/(q-\epsilon)'}, \tag{1}$$

where $A(\alpha, \epsilon)$ *is a constant depending only on* α *and* ϵ, *and, for* $2\leqq q\leqq\infty$,

$$M_q\left[\int_0^{2\pi} | x - t |^{\alpha/q'-1} d\mu(t) \right] \leqq A(\alpha) V_{1-\alpha}^{1/q'}, \tag{2}$$

where $A(\alpha)$ *is a constant depending only on* α.

We have

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$$\int_0^{2\pi} |x-t|^{\alpha/q'-1} d\mu(t) = \int_0^{2\pi} |x-t|^{-\alpha/q} dv_x(t),$$

where

$$v_x(t) = \int_0^t |x-s|^{\alpha-1} d\mu(s).$$

Consequently, using Hölder's inequality,

$$\left\{\int_0^{2\pi} |x-t|^{\alpha/q'-1} d\mu(t)\right\}^{q-\epsilon}$$
$$\leqq \left\{\int_0^{2\pi} |x-t|^{-\alpha(q-\epsilon)/q} dv_x(t)\right\} \left\{\int_0^{2\pi} dv_x(t)\right\}^{(q-\epsilon)/(q-\epsilon)'}$$

and this does not exceed

$$\left\{\int_0^{2\pi} |x-t|^{\alpha\epsilon/q-1} d\mu(t)\right\} V_{1-\alpha}^{(q-\epsilon)/(q-\epsilon)'}.$$

Thus, the left-hand side of (1) does not exceed

$$V^{1/(q-\epsilon)'} \left\{\int_0^{2\pi} d\mu(t) \int_0^{2\pi} |x-t|^{\alpha\epsilon/q-1} dx\right\}^{1/(q-\epsilon)},$$

which gives (1). It is surprising that so crude an argument gives a best possible result.

To prove (2) I first show the result true for $q=2$ and then that this implies its truth for $q>2$. For this latter portion of the proof I am indebted to Professor J. E. Littlewood.

We have first, inverting the order of integration,

$$M_2^2\left[\int_0^{2\pi} |x-t|^{\alpha/2-1} d\mu(t)\right]$$
$$= \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} |x-t|^{\alpha/2-1} |x-s|^{\alpha/2-1} dx\, d\mu(t)\, d\mu(s). \tag{3}$$

In the inner integral we make the substitution $x-t=(s-t)u$ and find that the integral does not exceed

$$|s-t|^{\alpha-1}\int_{-\infty}^{\infty} |u(1-u)|^{\alpha/2-1} du = B(\alpha)\,|s-t|^{\alpha-1}.$$

Consequently, the right-hand side of (3) is dominated by

$$B(\alpha)\int_0^{2\pi}\int_0^{2\pi} |s-t|^{\alpha-1}d\mu(t)d\mu(s) \leqq B(\alpha)V_{1-\alpha}(\mu(2\pi)-\mu(0)),$$

which gives the result for $q=2$.

For $q>2$, we have

$$M_q^q\left[\int_0^{2\pi} |x-t|^{\alpha/q'-1}d\mu(t)\right]$$
$$= \int_0^{2\pi} dx\left\{\int_0^{2\pi} |x-t|^{((q-2)/q)(\alpha-1)} |x-t|^{(\alpha-2)/q}d\mu(t)\right\}^q$$

and this, by Hölder's inequality, does not exceed

$$\int_0^{2\pi} dx\left\{\int_0^{2\pi} |x-t|^{\alpha-1}d\mu(t)\right\}^{q-2}\left\{\int_0^{2\pi} |x-t|^{\alpha/2-1}d\mu(t)\right\}^2,$$

which is, in turn, dominated by

$$V_{1-\alpha}^{q-2}M_2^2\left[\int_0^{2\pi} |x-t|^{\alpha/2-1}d\mu(t)\right] \leqq A(\alpha)V_{1-\alpha}^{q-1},$$

which gives the result for $q>2$.

2. **Proof of the theorem.** Suppose that E is a subset of $[0, 2\pi]$. If a nondecreasing $\mu(x)$ is such that

$$\int_E d\mu(t) = \int_0^{2\pi} d\mu(t) = 1,$$

we say that $\mu(x)$ is a distribution concentrated on E. If, further, for any β $(0<\beta<1)$ there is a $\mu(x)$ concentrated on E such that

$$V_\beta = \sup \int_0^{2\pi} |x-t|^{-\beta}d\mu(t) \quad \text{for all } x \in [0, 2\pi]$$

is finite, then E is said to be of positive β-capacity. Otherwise E is said to be of zero β-capacity. This definition is equivalent to that given by Salem and Zygmund [1].

Clearly, if E is of positive β-capacity, it is of positive γ-capacity for all $\gamma<\beta$. If it is of zero β-capacity, it is of zero γ-capacity for all $\gamma>\beta$.

We may, without loss of generality, assume $\int_0^{2\pi} f(x)dx=0$ and use the Weyl fractional integral.

Let $f(x)$ have the Fourier series

$$\sum_{n=-\infty}^{\infty}{}' c_n e^{inx}$$

where $'$ signifies that $c_0 = 0$. Then

$$f_{\alpha/q}(x) = \sum_{k=-\infty}^{\infty} (ik)^{-\alpha/q} c_k e^{ikx}$$

and it is sufficient to show that S_n is bounded outside a set of zero β-capacity, where $S_n = \sum_{-n}^{n} (ik)^{-\alpha/q} c_k e^{ikx}$ and $\beta = 1 - \alpha$ for $1 \leqq q \leqq 2$ and $\beta > 1 - \alpha$ for $q > 2$.

Assume then that S_n is unbounded in a set E of positive β-capacity. Then, first, there is a distribution $\mu(x)$ concentrated on E such that

$$\int_0^{2\pi} |x - t|^{-\beta} d\mu(t)$$

is bounded for all x. Secondly, by a well known argument there is a function $n(x) \leqq n$, taking only integer values, such that

$$\int_0^{2\pi} S_{n(x)}(x) d\mu(x)$$

exists and is unbounded as $n \to \infty$. I show this last to be impossible. For

$$\begin{aligned}\int_0^{2\pi} S_{n(x)}(x) d\mu(x) &= \int_0^{2\pi} \sum_{k=-n(x)}^{n(x)} (ik)^{-\alpha/q} c_k e^{ikx} d\mu(x) \\ &= \int_0^{2\pi} \int_0^{2\pi} f(t) \sum_{k=-n(x)}^{n(x)} (ik)^{-\alpha/q} e^{in(x-t)} dt d\mu(x).\end{aligned}$$

Now

$$\left| \sum_{k=-n(x)}^{n(x)} (ik)^{-\alpha/q} e^{ik(x-t)} \right| \leqq C |x - t|^{\alpha/q - 1}$$

so

$$\begin{aligned}\left| \int_0^{2\pi} S_{n(x)}(x) d\mu(x) \right| &\leqq C \int_0^{2\pi} |f(t)| \left\{ \int_0^{2\pi} |x - t|^{\alpha/q-1} d\mu(x) \right\} dt \\ &\leqq C M_q(f) M_{q'} \left[\int_0^{2\pi} |x - t|^{\alpha/q-1} d\mu(x) \right].\end{aligned}$$

Now $M_q(f) < \infty$ by hypothesis and we have only to show

(1) $$M_{q'}\left[\int_0^{2\pi} |x-t|^{\alpha/q-1}d\mu(x)\right]$$

bounded.

If $1 \leqq q \leqq 2$, then $q' \geqq 2$ and (2) of §1 immediately shows this.

If $q>2$, we write $\beta=1-\gamma$. Since $\gamma<\alpha$, there is an $r<q$ such that $\alpha/q=\gamma/r$. We may suppose β so near to $1-\alpha$ that $2<r<q$, since if we show the result for all such β it will immediately follow for all larger β. We may now rewrite (1) in the form

$$M_{r'-\epsilon}\left[\int_0^{2\pi} |x-t|^{\gamma/r-1}d\mu(x)\right],$$

which, since $r'<2$, is shown to be bounded by invoking (1) of §1. This gives the result.

3. **The theorem is best possible.** Construct the set S as follows. Let $\{\xi_n\}$ be any sequence such that $0<\xi_n<1/2$. From $S_0=[0, 2\pi]$ remove the open concentric interval of length $2\pi(1-2\xi_1)$, thus leaving the set S_1. From each of the intervals in S_1, of length $2\pi\xi_1$, remove an open concentric interval of length $2\pi\xi_1(1-2\xi_2)$, leaving a set S_2 consisting of four closed intervals each of length $2\pi\xi_1\xi_2$. Continuing in this way, we are left, after the kth removal, with a set S_k consisting of 2^k closed intervals each of length $2\pi\xi_1\xi_2 \cdots \xi_k$. Consequently $mS_k=2\pi 2^k\xi_1\xi_2 \cdots \xi_k$.

It is known [1, p. 40] that $S=\lim S_k$ will be of positive β-capacity if and only if

(1) $$\sum_{k=1}^{\infty} 2^{-k}(\xi_1\xi_2 \cdots \xi_k)^{-\beta} < \infty.$$

Define $\{f_n(x)\}$ on $[0, 2\pi]$ by

$$\begin{aligned} f_0(x) &= 0 \quad \text{in} \quad S_0, \\ f_n(x) &= (\xi_1\xi_2 \cdots \xi_n)^{-\alpha/q}n^{-1} \quad \text{in} \quad S_n \\ &= f_{n-1}(x) \quad \text{in} \quad S_0 - S_n \end{aligned}$$

for $n=1, 2, 3 \cdots$. Since $\{f_n(x)\}$ is an increasing sequence of measurable functions,

$$f(x) = \lim_{n\to\infty} f_n(x)$$

exists and is measurable.

It is easily seen that

$$f(x) = (\xi_1\xi_2 \cdots \xi_n)^{-\alpha/q} n^{-1} \text{ on } S_n - S_{n+1}, \qquad n = 1, 2, \cdots,$$

so that

$$\begin{aligned} \int_0^{2\pi} |f(x)|^q dx &= \sum_{n=1}^{\infty} \int_{S_n - S_{n+1}} |f(x)|^q dx \\ &= \sum_{n=1}^{\infty} (\xi_1\xi_2 \cdots \xi_n)^{-\alpha} n^{-q} [mS_n - mS_{n+1}] \\ &= \sum_{n=1}^{\infty} (1 - 2\xi_{n+1}) 2^n (\xi_1\xi_2 \cdots \xi_n)^{1-\alpha} n^{-q}. \end{aligned} \tag{2}$$

For $q > 2$ we may choose $\delta > 0$ so that $2(1+\delta) < q$ and then put

$$2\xi_n^{1-\alpha} = 1 + (1 + \delta)n^{-1}.$$

Then

$$2^{-k}(\xi_1\xi_2 \cdots \xi_k)^{\alpha-1} = O(k^{-1-\delta})$$

and so (1) with $\beta = 1 - \alpha$ is satisfied, showing S to be of positive $(1-\alpha)$ capacity.

Further (2) is clearly finite so that $f(x) \in L^q$.

Let x be any point of S. Now $S_k - S_{k+1}$ consists of 2^k intervals each of length $2\pi\xi_1\xi_2 \cdots \xi_k(1 - 2\xi_{k+1})$ none of which contain x. However, one of these intervals I_k is contained in an interval of S_k which contains x. Consequently

$$\int_0^{2\pi} |x - t|^{\alpha/q-1} f(t) dt = \sum_{k=1}^{\infty} \int_{S_k - S_{k+1}} \geqq \sum_{k=1}^{\infty} \int_{I_k}$$

and this last is itself greater than

$$\begin{aligned} (2\pi)^{\alpha/q} \cdot \sum_{k=1}^{\infty} (\xi_1\xi_2 \cdots \xi_k)^{\alpha/q-1} (\xi_1\xi_2 \cdots \xi_k)^{-\alpha/q} k^{-1} (\xi_1 \cdots \xi_k)(1 - 2\xi_{k+1}) \\ = (2\pi)^{\alpha/q} \sum_{k=1}^{\infty} (1 - 2\xi_{k+1}) k^{-1} = +\infty \end{aligned}$$

and so $\int_0^{2\pi} |x-t|^{\alpha/q-1} f(t) dt = +\infty$ at every point of S.

Now $f(x) = f(2\pi - x)$ so that

$$\int_0^{2\pi} |x - t|^{\alpha/q-1} f(t) dt = f_{\alpha/q}(x) + f_{\alpha/q}(2\pi - x)$$

where $f_{\alpha/q}$ now denotes the Riemann-Liouville fractional integral.

It follows that $f_{\alpha/q}(x)$ must be infinite in a set of positive $(1-\alpha)$-

capacity. For, if not, suppose $f_{\alpha/q}(x)$ infinite in a set M of zero $(1-\alpha)$-capacity. Then $f_{\alpha/q}(2\pi-x)$ is infinite in the mirror-image $\overline{M}$ of M about $x=\pi$. Also $M+\overline{M}=S$, and since both M and $\overline{M}$ are of zero $(1-\alpha)$-capacity, so is S. This contradiction gives the required result and shows part (b) of the theorem to be best possible.

Next, let β be any positive number less than $1-\alpha$ and let ξ be such that

$$2\xi^{(1-\alpha+\beta)/2} = 1.$$

Now consider the set S with $\xi_n=\xi$ for all n. Since $2\xi^\beta>1$, S is of positive β-capacity. Defining $f(x)$ as before we use exactly the same argument to show $f_{\alpha/q}(x)=+\infty$ in a set of positive β-capacity. Furthermore, since $2\xi^{1-\alpha}<1$, (2) is bounded, showing $f\in L^q$. This proves part (a) of the theorem to be best possible.

In passing, we may note that it has here been shown that a function in L^q for any q may be infinite in a set which is "only just" of measure zero. More precisely, given any $\beta<1$ and any $q>1$, there is a function in L^q which is infinite in a set of positive β-capacity, i.e., in a set of positive β-Hausdorff measure.

4. The lemma of §1 is best possible. Consider e.g., (2) of §1. Suppose this is not best possible, i.e., that there is an $\epsilon>0$ for which, in general,

$$M_{q+\epsilon}\left[\int_0^{2\pi} | x - t|^{\alpha/q'-1}d\mu(t)\right] < \infty.$$

If, then, $f(x)\in L^{(q+\epsilon)'}$, we may say

$$\left|\int_0^{2\pi} S_{n(x)}(x)d\mu(x)\right| \leqq CM_{(q+\epsilon)'}(f)M_{q+\epsilon}\left[\int_0^{2\pi} | x - t|^{\alpha/q'-1}d\mu(t)\right],$$

which is bounded. This would imply that (b) of the theorem is not best possible. Since it is, we have shown (2) best possible. A similar argument using (a) would show (1) best possible.

References

1. R. Salem and A. Zygmund, *Capacity of sets and Fourier series*, Trans. Amer. Math. Soc. vol. 59 (1946) pp. 23–41.

2. O. Frostman, *Potential d'équilibre et capacité des ensembles*, Meddelande från Lunds Univ. Mat. Sem. vol. 3 (1935).

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