ON THE ABSOLUTE SUMMABILITY OF THE CONJUGATE SERIES OF A FOURIER SERIES

T. PATI

1.1. Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. Let

$$\frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \tag{1.1.1}$$

be the Fourier series of $f(t)$. Then the conjugate series, or the allied series, of the Fourier series (1.1.1) is given by

$$\sum_{n=1}^{\infty} (b_n \cos nt - a_n \sin nt). \tag{1.1.2}$$

Let

$$\psi(t) = \frac{1}{2} \{f(x+t) - f(x-t)\}, \qquad \theta(t) = \frac{2}{\pi} \int_t^{\infty} \frac{\psi(u)}{u} du.$$

Suppose throughout $t > 0$. We write

$$\Psi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \psi(u) du, \qquad \alpha > 0,$$
$$\Psi_0(t) = \psi(t),$$

and

$$\psi_\alpha(t) = \Gamma(\alpha+1) t^{-\alpha} \Psi_\alpha(t), \qquad \alpha \geqq 0.$$

Similarly

$$\Theta_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1} \theta(u) du, \qquad \alpha > 0,$$
$$\Theta_0(t) = \theta(t),$$

and

$$\theta_\alpha(t) = \Gamma(\alpha+1) t^{-\alpha} \Theta_\alpha(t), \qquad \alpha \geqq 0.$$

1.2. The object of this paper is to study the nature of the conjugate series regarding its absolute summability.

In their paper *On the absolute summability of the allied series of a*

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Fourier series [2] Bosanquet and Hyslop established the following:

THEOREM A. *If $\alpha \geqq 0$, and $\psi_\alpha(t)$ is of bounded variation in* $(0, \pi)$ *and $\theta_\lambda(t)$ is of bounded variation in* $(0, \pi)$ *for some positive* λ, *then the allied series is summable* $|C, \alpha+\delta|$ *for every* $\delta > 0$.

By making use of an example they also proved that when $\alpha = 0$, this theorem is the best of its kind, that is to say, the theorem will fail when $\delta = 0$. In §2.1 I prove that also when $\alpha = 1$, the theorem will continue to remain the best of its kind.

1.3. It was shown by Whittaker [9] with the help of an example, suggested by Littlewood, that a Fourier series may converge at a point without being summable $|A|$ at that point, while Prasad [6] constructed an example of a series which is summable $|A|$ at a point without being convergent at that point. This shows that the properties of convergence and summability $|A|$ of infinite series are independent of each other. Fekete [4] proved that while summability $|C|$ implies summability $|A|$, the converse is not necessarily true. A natural question would be to examine the summability $|C|$ of Fourier series and conjugate series, when the series are not only summable $|A|$, but also convergent.

In §3.1 I show for Fourier series and conjugate series that summability $|A|$, even when coupled with convergence, does not necessarily imply summability $|C, 1|$ of these series.

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2.1. In this section I establish the following:

THEOREM 1. *If $\psi_1(t)$ is of bounded variation in* $(0, \pi)$ *and $\theta_\lambda(t)$ is of bounded variation in* $(0, \pi)$ *for some positive* λ, *then the conjugate series, although summable* $|C, 1+\delta|$ *for every* $\delta > 0$, *is not necessarily summable* $|C, 1|$.

2.2. I shall first prove that Theorem A of §1.2, for $\alpha = 1$, is equivalent to the following theorem of Bosanquet and Hyslop [2], for $\beta = 0$.

THEOREM B. *If $\beta \geqq 0$, and if*

$$(*) \qquad \int_0^\eta \frac{|\psi_\beta(t)|}{t} dt < \infty,$$

where $\eta > 0$, then the allied series is summable $|C, \beta+1+\delta|$ *at the point* $t = x$, *for every* $\delta > 0$.

I shall then take up an example for which the integral (*) for $\beta = 0$ exists, but the corresponding conjugate series is not summable $|C, 1|$.

2.3. In order to prove that Theorem A for $\alpha=1$ is equivalent to Theorem B for $\beta=0$, we require the following lemma.

LEMMA. *If $\alpha\geqq 0$, then the necessary and sufficient conditions that $\theta_\alpha(t)$ should be of bounded variation in $(0, \eta)$, where $\eta>0$, are that $\psi_{\alpha+1}(t)$ and $\theta_\lambda(t)$ should be of bounded variation in $(0, \eta)$ for some positive λ.*

For the proof of the lemma, reference may be made to Lemma 7 of the paper of Bosanquet and Hyslop.

In virtue of the lemma, when $\alpha\geqq 1$, the hypotheses in Theorem A are the necessary and sufficient conditions that $\theta_{\alpha-1}(t)$ should be of bounded variation in $(0, \pi)$. Now, since $\theta_{\alpha-1}(t)$ is an integral for $t>0$, it is a function of bounded variation in every interval (η, π), $0<\eta<\pi$. Hence, the hypotheses in Theorem A are equivalent to the bounded variation of $\theta_{\alpha-1}(t)$ in $(0, \eta)$, that is, to the convergence of the integral

$$\int_0^\eta \frac{|\psi_{\alpha-1}(t)|}{t}\,dt,$$

since, by a result due to Bosanquet [1], for $\alpha\geqq 1$, $(1/2)\pi\theta'_{\alpha-1}(t) = -\psi_{\alpha-1}(t)/t$, except possibly at a set of points of measure zero. The equivalence referred to in §2.2 follows immediately by taking $\alpha=1$.

2.4. Now let us consider the series

$$-\sum_{n=2}^{\infty}\frac{\sin n\alpha}{\log n}, \tag{2.4.1}$$

which is the conjugate series of the Fourier series

$$\sum_{n=2}^{\infty}\frac{\cos n\alpha}{\log n}, \tag{2.4.2}$$

it being known (Titchmarsh [8, §13.51]) that (2.4.1) is not itself a Fourier series.

The series (2.4.1) is convergent for all values of α. For $\alpha=\pi/2$, however, it is not absolutely convergent. Using the result (Kogbetliantz [5]) that if $\sum_1^\infty a_n$ is summable $|C, 1|$, then $\sum_1^\infty |a_n|/n$ is convergent, we see that (2.4.1) is not even summable $|C, 1|$ at $\alpha=\pi/2$.

Now, if $f(\alpha)$ be the sum-function of (2.4.2) for $0<\alpha<2\pi$, that is, the generating function of (2.4.2), then (Young [10])

$$\begin{aligned} f(\alpha) = &-\frac{1}{2}\cos\alpha-(\pi-\alpha)\sin\alpha \\ &+\frac{\pi}{2}\int_0^\infty \frac{\cosh y(\pi-\alpha)}{\sinh \pi y\{\pi^2/4+(\log y)^2\}}\,dy. \end{aligned}$$

Therefore, for $0 \leqq t < \pi/2$,

$$\psi(t) = \frac{1}{2}\left\{f\left(\frac{\pi}{2}+t\right) - f\left(\frac{\pi}{2}-t\right)\right\}$$

$$= \frac{1}{2}\sin t + t\cos t - \frac{\pi}{4}\int_0^\infty \frac{\sinh yt}{\cosh \frac{y\pi}{2}\left\{\frac{\pi^2}{4} + (\log y)^2\right\}}\, dy.$$

Hence,

$$|\psi(t)| \leqq \frac{1}{2}\sin t + t\cos t + \frac{\pi}{4}\int_0^\infty \frac{\sinh yt}{\cosh \frac{y\pi}{2}\left\{\frac{\pi^2}{4} + (\log y)^2\right\}}\, dy.$$

Let $0 < \eta < \pi/2$. Then

$$\int_0^\eta \frac{|\psi(t)|}{t}\, dt \leqq \frac{1}{2}\int_0^\eta \frac{\sin t}{t}\, dt + \int_0^\eta \cos t dt$$

$$+ \frac{\pi}{4}\int_0^\eta \frac{dt}{t}\int_0^\infty \frac{\sinh yt}{\cosh \frac{y\pi}{2}\left\{\frac{\pi^2}{4} + (\log y)^2\right\}}\, dy.$$

Now,

$$I \equiv \int_0^\eta \frac{dt}{t}\int_0^\infty \frac{\sinh yt}{\cosh \frac{y\pi}{2}\left\{\frac{\pi^2}{4} + (\log y)^2\right\}}\, dy$$

$$= \int_0^\infty \frac{dy}{\cosh \frac{y\pi}{2}\left\{\frac{\pi^2}{4} + (\log y)^2\right\}}\int_0^\eta \frac{\sinh yt}{t}\, dt,$$

if we change the order of integration by Fubini's theorem. Since $(\sinh yt)/t \leqq y \cosh yt$, we have $\int_0^\eta((\sinh yt)/t)dt \leqq \sinh y\eta$. This could also be inferred from term-by-term integration. Therefore,

$$I \leqq \int_0^\infty \frac{\sinh y\eta}{\cosh \frac{y\pi}{2}\left\{\frac{\pi^2}{4} + (\log y)^2\right\}}\, dy$$

$$= \frac{2}{\pi}\left\{\sin \eta + 2\eta\cos\eta + f\left(\frac{\pi}{2}-\eta\right) - f\left(\frac{\pi}{2}+\eta\right)\right\},$$

which is finite for every η, such that $0<\eta<\pi/2$. Also, since

$$\int_0^\eta \frac{\sin t}{t}\,dt < \infty,$$

we finally have, for $0<\eta<\pi/2$,

$$\int_0^\eta \frac{|\psi(t)|}{t}\,dt < \infty.$$

Thus, in the case of the conjugate series (2.4.1), at $\alpha=\pi/2$,

$$\int_0^\eta \frac{|\psi_0(t)|}{t}\,dt < \infty,$$

and still the series is not summable $|C, 1|$.

Combining the statements of §§2.2, 2.3, and 2.4, we obtain the theorem of §2.1.

3.1. For the case of the Fourier series, it can be demonstrated, as explained below, that summability $|A|$, coupled with convergence, does not necessarily imply summability $|C, 1|$.

Bosanquet and Kestelman [3] established that summability $|C, 1|$ of a Fourier series is not a local property. Thus, the bounded variation of $f(t)$ in the immediate neighbourhood of the point considered, $t=x$, is not sufficient to ensure the summability $|C, 1|$ of the corresponding Fourier series. On the other hand, we know from Jordan's criterion for the convergence of a Fourier series and from a theorem of Prasad [7] for the summability $|A|$ of Fourier series that bounded variation of $f(t)$ in the immediate neighbourhood of the point $t=x$ is sufficient to ensure the convergence and summability $|A|$, respectively, of the Fourier series of $f(t)$ at $t=x$. Thus, it is possible for a Fourier series to be both summable $|A|$ and convergent at a point without being necessarily summable $|C, 1|$ at that point.

For proving that even for conjugate series summability $|A|$, together with convergence, does not ensure summability $|C, 1|$ of the series, we shall make use of the example already mentioned in §2.4. The series (2.4.1) is summable $|A|$ at $\alpha=\pi/2$, since by a theorem of Prasad [7] the convergence of the integral

$$\int_0^\eta \frac{|\psi(t)|}{t}\,dt,$$

where $\eta>0$, is a sufficient condition for summability $|A|$. The series (2.4.1) is also convergent for all values of α. Yet, as has already been shown, it is not summable $|C, 1|$ at $\alpha=\pi/2$.

3.2. Thus, with the help of the example of the series (2.4.1), we obtain the following theorem.

THEOREM 2. *For the conjugate series of a Fourier series summability* $|A|$ *at a point, even when coupled with everywhere-convergence, does not necessarily imply summability* $|C, 1|$ *at that point.*

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UNIVERSITY OF ALLAHABAD