EQUATIONS EQUIVALENT TO A LINEAR DIFFERENTIAL EQUATION

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1. **Introduction.** Pinney [3] has remarked that the nonlinear equation $y''+qy=cy^{-3}$, where q is a function of the independent variable x and c is a constant, can be solved by the substitution $y^2=u^2-v^2$, where u, v are appropriately chosen solutions of the linear equation $u''+qu=0$. This suggests the question: what equations of order n have general solution expressible as $F(u_1, \cdots, u_n)$, where $u_1, \cdots, u_n$ constitute a variable set of solutions of a fixed linear differential equation? The present paper gives a partial answer to this question by determining all equations equivalent to linear equations (i) which are of the first order; (ii) which are homogeneous, of the second order, and have F depending on only one u; and (iii) which are homogeneous, of the second order, and have F homogeneous of nonzero degree in two u's.

Moreover, it is shown that:

The nonlinear equation

$$y'' - (\log w)'y' + kqy = (1 - l)y^{-1}y'^2 + cw^2y^{1-4l}, \quad kl = 1, \tag{1.1}$$

where c, k are constants and w, q are functions of the independent variable x, can be solved by putting

$$y^2 = u^kv^k, \quad c \neq 0; \qquad y = u^k, \quad c = 0, \tag{1.2}$$

where u, v satisfy the linear homogeneous equation

$$u'' - (\log w)'u' + qu = 0. \tag{1.3}$$

The function F giving the solution of (1.1) can be found by integrating a special equation of form (1.1) which has $w'=q=0$ and can be treated by elementary methods.

Pinney's result is got by making $k=w=1$ and replacing u, v by $u-v$, $u+v$.

Equations (1.1), (2.3), and (3.2) may represent new integrable types. Equation (1.1) resembles equation 6.53 of Kamke's collection [1, p. 554], but the fields of application merely overlap. If in (3.2) p, q are properly related, Kamke's 6.53 results.

For $c=0$, the result that equation (1.1) can be solved by the substitution (1.2) is Painlevé's [2, p. 35, equation (1)].

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2. **First order.** If u satisfies the linear nonhomogeneous equation

$$u' + pu + q = 0 \tag{2.1}$$

and $y = F(u)$, then

$$y' + (pu + q)F' = 0.$$

Set

$$F'(u) = f(y), \qquad u = \int f^{-1}dy.$$

The general form of first order equation integrable by the process of this paper is therefore

$$y' + \left(p\int f^{-1}dy + q\right)f = 0 \tag{2.2}$$

and the F is found by integrating

$$F' - f(F) = 0,$$

which is the special form assumed by (2.2) for $p = 0$, $q = -1$.

Any equation

$$y' + p(x)g(y) + q(x)f(y) = 0 \tag{2.3}$$

satisfying either of the conditions

$$f(gf^{-1})' = 1, \qquad g(fg^{-1})' = 1 \tag{2.4}$$

falls in this category.

For $f = y^n$, $n \neq 1$, equation (2.2) becomes Bernoulli's.

3. **Second order, F in one variable.** If

$$y = F(u), \qquad u'' + pu' + qu = 0,$$

then

$$y'' + py' = -uF'q + F''(F')^{-2}y'^2.$$

Setting

$$g(F) = uF' \tag{3.1}$$

gives

$$y'' + py' + qg(y) = [g'(y) - 1][g(y)]^{-1}y'^2. \tag{3.2}$$

The class sought consists of those equations which can be put in the form (3.2). For a given equation an F is found from (3.1). Note that

F also satisfies

$$F'' = [g'(F) - 1][g(F)]^{-1}F'^2, \tag{3.3}$$

the special form of (3.2) for $p = q = 0$.

If F is homogeneous of degree $k \neq 0$, then F can be taken as u^k and (3.2) assumes Painlevé's form [2, p. 35, equation (1)] which is also (1.1) for $c = 0$.

4. **Second order, homogeneous.** The problem involves eliminating u, v, u', v', u'', v'' among

$$\begin{gathered} y = F(u, v), \\ y' = u'F_u + v'F_v, \\ y'' = u''F_u + v''F_v + u'^2F_{uu} + 2u'v'F_{uv} + v'^2F_{vv}, \\ u'' + pu' + qu = 0, \qquad v'' + pv' + qv = 0. \end{gathered} \tag{4.1}$$

If we put

$$z = uF_u + vF_v, \quad w = uv' - u'v, \quad zu' = uy' - wF_v, \quad zv' = vy' + wF_u,$$

this operation is reduced to eliminating u, v between (4.1) and

$$y'' + py' = -qz + Ay'^2 + 2By'w + Cw^2, \tag{4.2}$$

where

$$\begin{gathered} z^2A = u^2F_{uu} + 2uvF_{uv} + v^2F_{vv}, \quad z^2C = F_v^2F_{uu} - 2F_uF_vF_{uv} + F_u^2F_{vv}, \\ z^2B = F_u(uF_{uv} + vF_{vv}) - F_v(uF_{uu} + vF_{uv}). \end{gathered} \tag{4.3}$$

By hypothesis, (4.2) is to reduce to $f(y'', y', y, p, q) = 0$, where y'', y', y, p, q are indeterminates. This entails that the right member of (4.2) reduce to a function of y by virtue of (4.1) when y', p, q are independently given arbitrary values. The indeterminate p can be replaced by w, subject to the restriction $w \neq 0$, since

$$w' + pw = 0. \tag{4.4}$$

Making $y' = q = 0$, $w = 1$ shows that C must be a function of F; $y' = 0$, $q = 1$ in the first three terms gives $z = z(F)$; $y' = 0$, $w = 1$ in the second and third divided by y' gives $B = B(F)$; and $y' = 1$ in the second gives finally $A = A(F)$.

The condition that z be a function of F is $B = 0$. For all such z

$$uF_{uu} + vF_{uv} = (z' - 1)F_u, \quad uF_{uv} + vF_{vv} = (z' - 1)F_v, \quad z' = dz/dF.$$

Direct substitution gives

$$(4.5)\qquad A = (z' - 1)z^{-1}, \quad B = 0, \quad C = (z' - 1)^{-1}z^{-1}(F_{uu}F_{vv} - F_{uv}^2),$$

the expression for C failing if $z'=1$ and those for A, C if $z=0$.

Now assume that F is homogeneous of degree k, where $k\neq 0$. Then $z=kF\neq 0$. The definition (4.3) shows that C is homogeneous of degree $k-4$. Setting

$$(4.6)\qquad F = u^k G(u^{-1}v)$$

and expressing the homogeneity of C give

$$C(m^k u^k G) = m^{k-4}C(u^k G).$$

Make $u=1$, replace m by u, replace v by $u^{-1}v$ and get

$$C(u^k G) = u^{k-4}C(G).$$

Evaluate for $u^{-1}v=a$, replace u by $[FG(a)^{-1}]^l$, where $kl=1$, and find

$$(4.7)\qquad C(F) = cF^{1-4l}.$$

Except for the constant c, which remains arbitrary, the equivalent equation (1.1) is completely determined.

To find F, seek G. From (4.6), (4.7)

$$C(F) = cu^{k-4}G^{1-4l}.$$

Substituting (4.6) in (4.3) gives

$$C(F) = u^{k-4}[G'' - (1 - l)G^{-1}G'^2].$$

Hence G is a solution of the equation

$$(4.8)\qquad G'' = (1 - l)G^{-1}G'^2 + cG^{1-4l},$$

a special case of (1.1) with $q=0$, $w=1$.

The independent variable does not appear explicitly in (4.8). By the usual elementary artifice that equation can be reduced to a linear equation of the first order in the dependent variable G'^2 and the independent variable G. It is sufficient here to note that

$$(4.9)\qquad G = (-4lc)^{k/4}(u^{-1}v)^{k/2}, \quad c \neq 0; \qquad G = 1, \quad c = 0$$

are particular solutions.

The case $c=0$ is straightforward. One variable serves, but another formula with two can also be obtained.

If $c\neq 0$, the constant appearing in (4.9) can be absorbed in u as it appears in F. To see this, suppose y given by (1.2) with u, v solutions of (1.3) whose Wronskian W has initial value W_0 satisfying

$$(4.10)\qquad W_0 = 2(-lc)^{1/2}w_0 \neq 0.$$

Then $W=2(-lc)^{1/2}w$. Either examination of the steps leading to (1.1) or direct substitution verifies that (1.1) is satisfied by such a y.

Initial values for u, v giving prescribed initial values y_0, y_0' and satisfying (4.10) are

$$\begin{aligned} u_0 &= y_0^{2l}, \\ u_0' &= l y_0^{2l-1} y_0' - (-lc)^{1/2} w_0, \\ v_0 &= 1, \\ v_0' &= l y_0^{-1} y_0' + (-lc)^{1/2} w_0 y_0^{-2l} \qquad (c \neq 0). \end{aligned} \tag{4.11}$$

If the above expressions for u_0, v_0, u_0', v_0' are multiplied respectively by a^{-1}, a, a^{-1}, a, where $a \neq 0$, the same values y_0, y_0', W_0 result. This corresponds to the fact that of the four constants in the pair u, v only three have been expended.

It will be noted that the constant c, when not zero, can be absorbed into w in (1.1), its role being simply to distinguish the two cases.

The case $k=0$ does not yield directly to the method of this paper. If $k=0$, then F is a function of a single variable $u^{-1}v$ but, contrary to what is true in §3, that variable does not satisfy a given linear equation.

References

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