ORDERED VECTOR SPACES

M. HAUSNER AND J. G. WENDEL

1. **Introduction.** An *ordered vector space* is a vector space V over the reals which is simply ordered under a relation $>$ satisfying:

(i) $x>0$, λ real and positive, implies $\lambda x>0$;
(ii) $x>0$, $y>0$ implies $x+y>0$;
(iii) $x>y$ if and only if $x-y>0$.

Simple consequences of these assumptions are: $x>y$ implies $x+z>y+z$; $x>y$ implies $\lambda x>\lambda y$ for real positive scalars λ; $x>0$ if and only if $0>-x$.

An important class of examples of such V's is due to R. Thrall; we shall call these spaces *lexicographic function spaces* (LFS), defining them as follows:

Let T be any simply ordered set; let f be any real-valued function on T taking nonzero values on at most a well ordered subset of T. Let V_T be the linear space of all such functions, under the usual operations of pointwise addition and scalar multiplication, and define $f>0$ to mean that $f(t_0)>0$ if t_0 is the first point of T at which f does not vanish. Clearly V_T is an ordered vector space as defined above. What we shall show in the present note is that every V is isomorphic to a subspace of a V_T.

2. **Dominance and equivalence.** A trivial but suggestive special case of V_T is obtained when the set T is taken to be a single point. Then it is clear that V_T is order isomorphic to the real field. As will be shown later on, this example is characterized by the *Archimedean property*: if $0<x$, $0<y$ then $\lambda x<y<\mu x$ for some positive real λ, μ.

Returning to the general case let V be any ordered vector space, and V^+ its set of positive elements. It is convenient to have a notation to indicate failure of the Archimedean property, as follows. Let $x, y\in V^+$. If $\lambda x<y$ for all positive real λ, we say that *x is dominated by y* and write $x\ll y$, or $y\gg x$. Clearly the relation $\ll$ is nonreflexive, nonsymmetric, and transitive; and $x\ll y$ implies $x<y$.

For given $x, y\in V^+$, if neither of x and y dominates the other we say that *x and y are equivalent*, and write $x\sim y$. This relation is characterized by the existence of positive real λ, μ such that $\lambda x<y<\mu x$, and it follows that it is indeed an equivalence relation on V^+. We denote the class of elements of V^+ equivalent to given x by $[x]$.

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Now we observe that there is a natural ordering on the set of equivalence classes; we define $[x]<[y]$ to mean that $x \gg y$. This definition is easily justified by the observation that if $x \sim x'$, $y \sim y'$, and $x \gg y$, then $x' \gg y'$. Our notation may be somewhat confusing; however, $[x]<[y]$ is to be thought of as meaning, roughly "$[x]$ comes before, or is more important than, $[y]$." An expression of frequent occurrence in the sequel is $[x] \leqq [y]$; this means that either x dominates or is equivalent to y—hence that *y does not dominate x.*

In the case where V is an LFS, say $V=V_T$, it is easy to discern the meanings of dominance and equivalence. In fact, $f_1 \gg f_2$ means that f_1 fails to vanish before f_2 does. More precisely, if t_i is the first t for which $f_i(t) \neq 0$, then $t_1<t_2$. From this it follows that $f_1 \sim f_2$ if and only if $t_1=t_2$, and that $[f_1]<[f_2]$ if and only if $t_1<t_2$. In other words, the ordered set of equivalence classes is order isomorphic to the underlying set T.

LEMMA 2.1. *If $x \gg x_1, x_2, \cdots, x_n$ and $\lambda, \lambda_1, \lambda_2, \cdots, \lambda_n$ are positive real numbers, then*

$$\lambda x+\lambda_1 x_1+\lambda_2 x_2+\cdots+\lambda_k x_k \gg \lambda_{k+1} x_{k+1}+\cdots+\lambda_n x_n.$$

PROOF. We have $x>(n\mu\lambda_i/\lambda)x_i$ for all real $\mu>0$; therefore $(\lambda/n)x > \mu\lambda_i x_i$, and $\lambda x>\mu(\lambda_{k+1}x_{k+1}+\cdots+\lambda_n x_n)$. Hence

$$\lambda x+\lambda_1 x_1+\lambda_2 x_2+\cdots+\lambda_k x_k>\mu(\lambda_{k+1} x_{k+1}+\cdots+\lambda_n x_n),$$

all positive real μ, which is what we had to show.

COROLLARY 2.2. *If $\{x_t\}$ is a set of elements of V^+ no two of which are equivalent, then the x_t are linearly independent.*

PROOF. If there is linear dependence among the x_t, we shall obtain an equation of the form

$$\lambda x+\lambda_1 x_1+\lambda_2 x_2+\cdots+\lambda_k x_k=\lambda_{k+1} x_{k+1}+\cdots+\lambda_n x_n,$$

where all λ's are positive and real, all x's belong to the given set, and x dominates $x_1, x_2, \cdots, x_n$. But, this, in view of Lemma 2.1, is a contradiction.

Before stating the next lemmas it is convenient to introduce the notion of absolute value, defined by: $|x|=x, -x$, or 0 according as $x \in V^+$, $-x \in V^+$, or $x=0$. Clearly the triangle inequality $|x+y| \leqq |x|+|y|$ and the multiplicative relation $|\lambda x|=|\lambda||x|$ hold, for $x, y \in V$ and real λ.

LEMMA 2.3. *If $[|x-y|] \leqq [|x-z|]$, then $[|x-y|] \leqq [|y-z|]$.*

PROOF. We are given that $|x-z|$ does not dominate $|x-y|$, and must prove that $|y-z|$ does not dominate $|x-y|$. But if $|y-z| > \lambda|x-y|$ for all real λ, then $\lambda|x-y| < |y-z| \leqq |x-y| + |x-z|$ for all λ, so that $(\lambda-1)|x-y| < |x-z|$ for all λ, which contradicts the assumption.

LEMMA 2.4. *If $x \sim |y|$, there is a unique λ such that $\lambda x = y$ or $|\lambda x - y| \ll x$.*

PROOF. The uniqueness of λ is immediate, for if $\lambda_1 \neq \lambda_2$, we have $|(\lambda_1-\lambda_2)x| \leqq |\lambda_1 x - y| + |\lambda_2 x - y|$, and if both terms on the right were zero or dominated by x, we should have $x \ll x$.

To show that one such λ exists we have $\mu x < |y| < \nu x$ for some positive real μ, ν. Let λ' be the supremum of the numbers μ for which $\mu x < |y|$. Then for $\epsilon > 0$ we have $(\lambda' - \epsilon)x < |y| < (\lambda' + \epsilon)x$; therefore $-\epsilon x < |y| - \lambda' x < \epsilon x$. Take $\lambda = \lambda'$ or $-\lambda'$ according as y is positive or negative; changing signs if necessary we have $-\epsilon x < y - \lambda x < \epsilon x$. Therefore either $y - \lambda x = 0$, or $|y - \lambda x| \ll x$; this completes the proof.

Sufficient machinery is now at hand for the investigation of structure questions. The finite-dimensional case is very easy; although the result is known [1, p. 240] we give it here as an illustration of the method.

THEOREM 2.5. *Let V be a finite-dimensional ordered vector space. A basis $(e_1, e_2, \cdots, e_n)$ can be chosen so that the ordering in V is lexicographic, i.e.,*

$$x = \sum_{i=1}^{n} \lambda_i e_i > 0$$

if and only if the first nonvanishing λ_i is positive. In other words, V is the lexicographic function space V_T on the ordered set $T = (1, 2, \cdots, n)$.

PROOF. Let V^+ be decomposed into equivalence classes as above, and for each equivalence class t let e_t be an arbitrary element of it; by Corollary 2.2 the set $\{e_t\}$ is finite. That is, $T = \{t\}$ is a finite set, and we may choose the notation so that $T = \{1, 2, \cdots, k\}$ with $e_1 \gg e_2 \gg \cdots \gg e_k$; clearly k does not exceed $n = \dim V$.

Let $y \in V$. If $y \neq 0$, then $|y|$ belongs to some equivalence class, say $|y| \sim e_{t_1}$. Applying Lemma 2.4 there is a unique λ_1 such that $|y - y_1 e_{t_1}| \ll e_{t_1}$, or $y = \lambda_1 e_{t_1}$. We may now repeat the process on $y - \lambda_1 e_{t_1}$, if it is not zero, and so on until the zero element is reached, as it must be in a finite number of steps. Thus we see that y is indeed a linear combination of the e_t; since y was arbitrary, it follows that the e_t constitute a basis for V, and, moreover, that $k = n$. This completes the proof.

COROLLARY 2.6. *If V has the Archimedean property, then* dim $V=1$.

PROOF. There is only one equivalence class.

It should be pointed out that there is a high degree of arbitrariness in the choice of basis for a finite-dimensional V. In fact, if A is any lower triangular matrix with positive diagonal elements, then the equation $Ae_i=e_i'$ carries the basis $(e_1, e_2, \cdots, e_n)$ into another lexicographic basis $(e_1', e_2', \cdots, e_n')$. Conversely, any two bases are connected by a transformation of this form.

3. The embedding theorem for general V. It is evident that no such simple structure theorem will hold for arbitrary infinite-dimensional ordered vector spaces. For example, let T be the set of positive integers in their natural ordering and form the lexicographic function space V_T. We get just the space of all real sequences, whose dimension as a vector space is the power of the continuum. But no vector space basis can be lexicographic in the sense of §2, for the set of equivalence classes is in 1–1 correspondence with the points of T and therefore is a countable set. A slight modification of this example shows that, moreover, not every V is an LFS. Let V be the subset of V_T consisting of finite linear combinations of characteristic functions f_t, $t\in T$, where $f_t(s)=0$ or 1 according as s differs from or equals t. The set of equivalence classes of V is again isomorphic to T, so that if V were an LFS, it would have to be isomorphic to V_T; but this is impossible since the dimension of V is $\aleph_0$.

The truth lies somewhere between these extremes; we shall show that associated with any V there is a unique V_T such that V is isomorphic to a "large" subspace of V_T. Before stating the precise result we need a definition.

Let V_T be an LFS, let $t_0\in T$, and let C be the linear transformation which truncates every $f\in V_T$ at t_0—that is, $Cf=g$, where $g(t)=f(t)$ for $t<t_0$ and $g(t)=0$ for $t\geqq t_0$. We shall call C the *cut* determined by t_0.

THEOREM 3.1. *Let V be an ordered vector space, let T be the set of equivalence classes of V^+, and for each $t\in T$ let a representative vector $e_t\in t$ be selected. Form the space V_T, denoting the characteristic function of the point t by f_t. There is a mapping F of V to V_T satisfying the following requirements*:

(i) *F is linear*;

(ii) *F is* 1–1;

(iii) *F is order preserving*;

(iv) *$F(e_t)=f_t$, $t\in T$*;

(v) If *$f\in F(V)$ and C is any cut, then $Cf\in F(V)$*.

This theorem has to be proved by a nonconstructive method. As a first step in the transfinite induction process we have:

THEOREM 3.2. *Let* V_0 *be a proper subspace of* V *which is mapped into* V_T *by a function* F: $y \to y'$ *satisfying* (i)–(v) *above. Let* $x \notin V_0$, *and let* V_1 *be the subspace spanned by* x *and* V_0. *Then there is an extension of the mapping* F *having domain* V_1 *and again satisfying* (i)–(v). (*We are assuming that* (iv) *is not vacuously satisfied*; *in other words that* V_0 *contains all of the* e_t.)

PROOF. Let S be the set of equivalence classes $[|x-y|]$ for $y \in V_0$. We observe that S has no last element. In fact, suppose that $[|x-y|] \leqq [|x-z|]$ for some $z \in V_0$ and all y. Let t be the equivalence class to which $|x-z|$ belongs; we have $|x-z| \sim e_t$, and therefore by Lemma 2.4 there is a constant λ such that either $x-z=\lambda e_t$ or $|x-z-\lambda e_t| \ll e_t$. Since $z+\lambda e_t$ is again an element of V_0, both alternatives yield contradictions and we have the result.

Let R be a well-ordered subset of S which is cofinal in S, so that for $[|x-y|] \in S$ there is an $[|x-z|] \in R$ such that $[|x-y|] < [|x-z|]$. We index the elements of R by ordinals α less than some limit ordinal θ, obtaining $R = \{[|x-z_\alpha|]\}$, where $\alpha < \beta$ implies $[|x-z_\alpha|] < [|x-z_\beta|]$. For each $\alpha < \theta$ let t_α denote the equivalence class $[|x-z_\alpha|]$; then $t_\alpha < t_\beta$ for $\alpha < \beta$.

From Lemma 2.3 it follows that $t_\alpha = [|x-z_\alpha|] \leqq [|z_\alpha - z_\beta|]$, and therefore e_{t_α} is not dominated by $|z_\alpha - z_\beta|$ for $\beta > \alpha$. Applying the mapping F we find that $F(e_{t_\alpha}) = f_{t_\alpha}$ is not dominated by $|F(z_\alpha) - F(z_\beta)| = |z'_\alpha - z'_\beta|$. Therefore $z'_\alpha(t) - z'_\beta(t) = 0$ for $t < t_\alpha$. We can now define the function x' which is to be the image in V_T, under the extension of F, of the given element x. For any $t \in T$ which is less than some $t_\alpha \in R$ let $x'(t) = z'_\alpha(t)$, and for the remaining $t \in T$ set $x'(t) = 0$. This definition is legitimate, for if $t < t_\alpha$ and also $t < t_\beta$, then $z'_\alpha(t) = z'_\beta(t)$ and the function x' so defined clearly vanishes except at the points of a well-ordered set.

The mapping F is now extended to all of $V_1 = \{\lambda x + y \mid y \in V_0, \lambda \text{ real}\}$ by defining $F(\lambda x + y) = \lambda x' + y'$; we shall verify that F on V_1 has the properties (i)–(v).

The requirements (i) and (iv) are immediately seen to hold. In order to prove (v) let C be any cut, and let $f \in F(V_1)$. The element f has the form $f = \lambda x' + xy'$, and therefore $Cf = \lambda Cx' + Cy'$. If the $t_0 \in T$ which determines C is less than one of the t_α, then $Cx' = Cz'_\alpha$ and $Cf = C(\lambda z'_\alpha + y')$ which by hypothesis is the cut of some element of V_0. If t_0 exceeds all t_α, then $Cx' = x'$, $Cf = \lambda x' + Cy' = \lambda x' + y'_1$ for some $y_1 \in V_0$, so that $Cf = F(\lambda x + y_1)$.

We next show that the extension is 1–1; it is enough to prove that $x' \neq y'$ for $y \in V_0$. Supposing the contrary let $y' = x'$ for some $y \in V_0$. Then $y'(t) = z'_\alpha(t)$ for $t < t_\alpha$, and therefore $|y' - z'_\alpha|$ does not dominate f_{t_α}. Since F preserves order on V_0, this implies that $|y - z_\alpha|$ does not dominate e_{t_α}. Hence $t_\alpha = [|x - z_\alpha|] \leqq [|y - z_\alpha|]$. Applying Lemma 2.3 we have $[|x - z_\alpha|] \leqq [|x - y|]$ for all α. But R was cofinal in S and S has no last element; this contradiction yields the result. As a corollary to this we can state the following. Let W be the set of t at which x' does not vanish. Let $y \in V_0$ and let Y be the set of t at which y' does not vanish. If there are any $t \in Y$ which exceed all of the points of W, let t_0 be the least such. Then, it is not the case that $y'(t) = x'(t)$ for all $t < t_0$. For otherwise, let C be the cut determined by t_0 and apply C to y'. We have $Cy' = x'$, but Cy' is the image of some $y_1 \in V_0$.

Finally we have to show that the extension of F preserves order. Let $x > y$, $y \in V_0$, and suppose that $x' < y'$. (We already know that $x' \neq y'$.) Let t_0 be the first point at which $x'(t) \neq y'(t)$. We have $x'(t_0) < y'(t_0)$, and by the corollary just proved t_0 does not exceed all of the points of W. Hence there is a $t_\alpha \in R$ such that $t_0 < t_\alpha$. $x'(t_0) = z'_\alpha(t_0) < y'(t_0)$, but $z'_\alpha(t) = y'(t)$ for $t < t_0$. Therefore $z'_\alpha < y'$. Since F on V_0 is order preserving we have $z_\alpha < y$. Then $x > y > z_\alpha$, so that $y - z_\alpha < x - z_\alpha$ and $x - z_\alpha$ is not dominated by $y - z_\alpha$. Therefore $t_\alpha = [|x - z_\alpha|] \leqq [|y - z_\alpha|]$. But then $y'(t) = z'_\alpha(t)$ for $t < t_\alpha$, and in particular for $t = t_0$.

This contradiction shows that $x > y$ implies $x' > y'$. In a similar way we can show that $x < y$ implies $x' < y'$. Therefore, if $\lambda x + y > 0$ with $\lambda > 0$ we have $x > -y/\lambda$, $x' > -y'/\lambda$, and so $\lambda x' + y' > 0$; a similar calculation yields the result for negative λ. This completes the proof of (iii) and of the theorem.

Proof of Theorem 3.1. F's satisfying the hypotheses of Theorem 3.2 surely exist, since we may take, for example, V_0 equal to the span of the e_t and define F to be the linear extension of the function defined by (iv), the e_t being linearly independent by Corollary 2.2. We partially order the set of all such mappings by the definition $F_1 < F_2$ if F_2 is a proper extension of F_1. Clearly the hypotheses of Zorn's lemma are fulfilled, and there is a maximal F. The domain of F is all of V, for otherwise by Theorem 3.2 F has a proper extension. Q.E.D.

Reference

1. Garrett Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloquium Publications, vol. 25, rev. ed., New York, 1949.

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