

REMARK ON A THEOREM OF FROBENIUS¹

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1. Introduction. A finite group G will be called a Frobenius group with respect to one of its subgroups H if:

- a. H is its own normalizer (is *ipsonormal*) in G ; and
- b. the intersection of H with any of its conjugates in G is the identity element E .

It is an unsolved problem of group theory to prove, without using group characters, the following theorem of Frobenius: If G is a Frobenius group with respect to H , then the set N consisting of the identity element and those elements of G not in H or its conjugates is a normal subgroup of G . The purpose of this note is to establish the theorem for solvable H , without using characters but instead by applying a transfer.

2. Application of transfers. If g is an element of a group G , then the mapping

$$g \rightarrow V_{G \rightarrow H}(g) = H' \prod_{i=1}^r g_i g [(g_i g)^*]^{-1}$$

(where $G = H \cup Hg_1 \cup \dots \cup Hg_r$, H' is the first derived group of H , and x^* denotes the representative of the right coset containing x in the given decomposition) carries G into H/H' and is called the transfer of G into H . It is shown in Chapter V of Zassenhaus² that transfer is independent of the choice of representatives and that transfer is homomorphic. These properties are easily established by direct calculation not involving matrices because of the fact that commutator quotient groups are abelian.

In view of this it is now possible to establish the following theorem.

THEOREM. *If G is a Frobenius group with respect to a solvable subgroup H , then N is a normal subgroup of G .*

PROOF. Let H be k -step metabelian. Decompose G into right cosets of H :

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² Hans Zassenhaus, *The theory of groups* (trans. by Saul Kravetz), New York, Chelsea, 1949.

$$G = H + Hg_2 + Hg_3 + \dots + Hg_r.$$

Transfer G to H by the mapping V_1 . Consider any element h of H . The factor contributed to $V_1(h)$ by the coset H is $EhE^{-1} = h$. So $V_1(h) = H'h \prod_{i=1}^s g_{a_i} h^{p_i} g_{a_i}^{-1}$. But $g_{a_i} \notin H$, and $g_{a_i} h^{p_i} g_{a_i}^{-1} \in H$. Therefore $g_{a_i} h^{p_i} g_{a_i}^{-1} = E$, and $V_1(h) = H'h$ for all $h \in H$. In particular, $V_1(E) = H'$, and $V_1(h') = H'$ for all $h' \in H'$.

Now consider an element of a conjugate of H , say $m^{-1}hm$, $h \in H$, $m \in G - H$. The factor contributed to $V_1(m^{-1}hm)$ by the coset Hm is $mm^{-1}hmm^{-1} = h$. So

$$V_1(m^{-1}hm) = H'h \prod_{i=1}^t g_{a_i} (m^{-1}h^{q_i}m) g_{a_i}^{-1}.$$

But $g_{a_i} \notin Hm$. Therefore every $g_{a_i} (m^{-1}h^{q_i}m) g_{a_i}^{-1} = E$, and $V_1(m^{-1}hm) = H'h$ for all $h \in H$, $m \in G - H$. For those elements $h' \in H'$,

$$V_1(m^{-1}h'm) = H'.$$

Now consider an element n of the set N consisting of E and those elements of G not in H or its conjugates. Take $n \neq E$.

$$V_1(n) = H' \prod_{i=1}^w g_{a_i} n^{r_i} g_{a_i}^{-1}.$$

Each factor $g_{a_i} n^{r_i} g_{a_i}^{-1}$ lies in H , by construction.

But every power of an element of N , except for a multiple of the order of that element, lies in N , but not in H or its conjugates; and all conjugates of elements of N lie in N .

Hence each factor is the identity, and

$$V_1(n) = H' \quad \text{for all } n \in N.$$

Now V_1 defines a homomorphism of G onto H/H' . Its kernel is the group $Q_1 \subset G$:

$$Q_1 = N + \sum_{m \in G} m^{-1}H'm, \quad \text{each element counted once.}$$

If $k=1$, then $H'=E$, and the theorem is proved.

If $k>1$, $H' \supset H''$, and so $Q_1 \supset H''$.

Transfer Q_1 to H' by the mapping V_2 . By exactly similar argument, this gives rise to the group

$$Q_2 = N + \sum_{m \in Q_1} m^{-1}H''m, \quad \text{each element counted once,}$$

and Q_2 is a subgroup of Q_1 and a fortiori of G .

But H is solvable and finite, and so k is finite. At length, by iteration, we arrive at

$$Q_k = N,$$

which is therefore a subgroup of G . Since N is a normal set in G , the theorem is proved.

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