ON THE INTEGRAL EQUATION $\lambda f(x) = \int_0^a K(x-y)f(y)dy$

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1. **Introduction.** We wish to consider the integral equation

$$\lambda f(x) = \int_0^a K(x-y)f(y)dy, \qquad a > 0, \tag{1}$$

which occurs in connection with various problems of probability theory and mathematical physics. Unless $K(x)$ is a function of particularly simple type, such as a polynomial or sum of exponentials, the problem of obtaining an exact solution of (1) appears exceedingly difficult. In the present note we discuss the behavior of the largest characteristic value, λ_M, as $a \to \infty$, under certain assumptions concerning $K(x)$, and illustrate our results with reference to the integral equation of Kac,

$$\lambda f(x) = \int_0^a e^{-(x-y)^2} f(y)dy. \tag{2}$$

The principal result is

THEOREM 1. *If*

$$\begin{aligned} &\text{(a) } K(x) \textit{ is non-negative, even, and monotone decreasing} \\ &\qquad \textit{for } 0 \leqq x < \infty, \\ &\text{(b) } c = \int_0^\infty K(x)dx < \infty, \end{aligned} \tag{3}$$

then as $a \to \infty$, $\lambda_M \to 2c$.

More precisely, for all $a > 0$,

$$2\int_0^{a/2} K(x)dx \geqq \lambda_M \geqq 2\int_0^a K(x)dx - \frac{2}{a}\int_0^a xK(x)dx. \tag{4}$$

Our first method of proof depends upon two tools, the classical Rayleigh-Ritz procedure and a new variational procedure introduced by Bohnenblust. The second method utilizes some known techniques of the theory of integral equations, and exhibits an important property of the characteristic function associated with λ_M.

2. **First proof.** We shall employ the following two lemmas, the first of which is well known:

Received by the editors June 18, 1951 and, in revised form, February 25, 1952.

LEMMA 1. *If $K(x, y)$ is real, symmetric, and satisfies the condition that*

$$\int_0^a \int_0^a K^2(x, y)dxdy < \infty,$$

then

$$\lambda_M = \operatorname*{Max}_f \frac{\int_0^a \int_0^a K(x, y)f(x)f(y)dxdy}{\int_0^a f^2(x)dx}. \tag{1}$$

LEMMA 2. *If $K(x, y)$ is bounded and non-negative for $0 \leqq x, y \leqq a$, and λ_M denotes, as above, the largest characteristic value of $K(x, y)$, then*

$$\sup_{g \geqq 1} \operatorname*{Min}_x \frac{\int_0^a K(y, x)g(y)dy}{g(x)} \leqq \lambda_M \leqq \inf_{g \geqq 1} \operatorname*{Max}_x \frac{\int_0^a K(y, x)g(y)dy}{g(x)}. \tag{2}$$

PROOF OF LEMMA 2. As is known, the characteristic function associated with λ_M may be taken to be positive, by virtue of the non-negativity of $K(x, y)$, taking K to be nontrivial. Let $g(x)$ be a positive function greater than or equal to one. From

$$\lambda_M f(x) = \int_0^a K(x, y)f(y)dy, \tag{3}$$

we obtain

$$\begin{aligned} \lambda_M \int_0^a f(x)g(x)dx &= \int_0^a \left(\int_0^a K(x, y)g(x)dx\right) f(y)dy \\ &= \int_0^a \frac{\left(\int_0^a K(x, y)g(x)dx\right)}{g(y)} f(y)g(y)dy \end{aligned} \tag{4}$$

whence (2) follows immediately. That the two sides of the inequality in (2) are actually equal and equal to λ_M is a result of Bohnenblust.

Lemma 1 contains the essence of the Rayleigh-Ritz method and furnishes lower bounds for λ_M. Lemma 2, which is also based upon variational principles, furnishes upper and lower bounds. Combining the two, and using the fact that $K(x)$ is even, we obtain

$$\inf_{g \geqq 1} \max_{0 \leqq x \leqq a} \frac{\int_0^a K(x-y)g(y)dy}{g(x)} \geqq \lambda_M$$

(5)
$$= \max_f \frac{\int_0^a \int_0^a K(x-y)f(x)f(y)dxdy}{\int_0^a f^2(x)dx}$$

$$\geqq \sup_{g \geqq 1} \min_{0 \leqq x \leqq a} \frac{\int_0^a K(x-y)g(y)dy}{g(y)}.$$

The simplest possible choices of f and g, viz., $f=g=1$, yield (4) of §1. It is clear that these results may be further refined by a cleverer choice of f and g. However, the calculations rapidly become complicated.

Setting $f=1$, we obtain

(6)
$$\lambda_M \geqq \int_0^a \left[\int_0^a K(x-y)dy\right] dx/a$$
$$= \frac{1}{a}\int_0^a \left[\int_0^x K(u)du + \int_0^{a-x} K(u)du\right] dx$$
$$= \frac{2}{a}\int_0^a \left[\int_0^x K(u)du\right] dx.$$

Integration by parts yields

(7)
$$\lambda_M \geqq 2\int_0^a K(u)du - \frac{2}{a}\int_0^a uK(u)du.$$

Setting $g=1$, we obtain

(8)
$$\max_{0 \leqq x \leqq a} \int_0^a K(x-y)dy \geqq \lambda_M.$$

Since K is even and monotone decreasing, it is easily seen that the maximum occurs at $x=a/2$. Thus,

$$\int_0^a K\left(\frac{a}{2}-y\right)dy = 2\int_0^{a/2} K(y)dy \geqq \lambda_M. \tag{9}$$

If $\int^\infty K(x)dx < \infty$, it follows readily that $\int_0^a xK(x)dx = o(a)$ as $a\to\infty$, and thus $\lambda_M \to 2\int_0^\infty K(u)du$ as $a\to\infty$.

The bounds for λ_M obtained in this way will be narrow only for fairly large a, the magnitude depending upon $K(x)$. Taking the Kac case, $K(x)=e^{-x^2}$, we obtain

$$2\int_0^{a/2} e^{-x^2}dx \geqq \lambda_M \geqq 2\int_0^a e^{-x^2}dx - \frac{1}{a} + \frac{e^{-a^2}}{a} \tag{10}$$

which yields the results

$$\begin{aligned} .843 &\geqq \lambda_M(2)/\pi^{1/2} \geqq .713, \\ .995 &\geqq \lambda_M(4)/\pi^{1/2} \geqq .749, \\ .999 &\geqq \lambda_M(10)/\pi^{1/2} \geqq .899. \end{aligned} \tag{11}$$

Notice that even for small a, (10) yields a rough idea of the true value of λ_M.

3. **Second proof.** The method we present below yields the following useful result:

THEOREM 2. *If $K(x)$ is non-negative, continuous, even, and monotone decreasing for $0\leqq x<\infty$, the characteristic function $f_M(x)$ associated with λ_M, which we normalize by the requirement that $\int_0^a f_M(x)dx=1$, possesses the following properties*:

$$\begin{aligned} &\text{(a) } f_M(x) = f_M(a-x), \\ &\text{(b) } f_M \textit{ is monotone increasing in } 0 \leqq x \leqq a/2. \end{aligned} \tag{1}$$

PROOF. We require the following two lemmas, the first of which is a well known result in the theory of integral equations:

LEMMA 3. *Let $K(x, y)$ be a continuous symmetric function defined over the square $0\leqq x, y\leqq a$, and $g(x)$ be continuous over $0\leqq x\leqq a$. Then, if we define*

$$Tg = \int_0^a K(x, y)g(y)dy, \tag{2}$$

the limit

$$\lim_{n\to\infty} \frac{T^n g}{\lambda_M^n} = \phi(x) \tag{3}$$

exists and is a characteristic function of $K(x, y)$ associated with λ_M, provided that it is not identically zero.

LEMMA 4. *If $f(x)$ has the following properties:*

$$\text{(4)} \qquad \begin{aligned} &\text{(a)}\ \ f(x) = f(a-x),\\ &\text{(b)}\ f'(x) \geqq 0 \quad \text{for} \quad 0 \leqq x \leqq a/2,\\ &\text{(c)}\ \ f(0) \geqq 0, \end{aligned}$$

then

$$\text{(5)} \qquad Tf = \int_0^a K(x-y)f(y)dy$$

possesses the same properties, provided that $K(x)$ is even, non-negative, monotone decreasing in the interval $[0, a]$, and possesses a derivative in this interval.

PROOF OF LEMMA 4. We have

$$\text{(6)} \qquad g(x) = Tf = 2\int_0^{a/2} [K(x-y) + K(a-x-y)]f(y)dy$$

whence

$$\text{(7)} \qquad g'(x) = 2\int_0^{a/2} [K'(x-y) - K'(a-x-y)]f(y)dy.$$

Integration by parts yields

$$\text{(8)} \qquad \begin{aligned} g'(x) &= 2f(0)[K(x) - K(a-x)]\\ &\quad + 2\int_0^{a/2} [K(x-y) - K(a-x-y)]f'(y)dy. \end{aligned}$$

If $0 \leqq x,\ y \leqq a/2$, we have

$$x \leqq a-x, \quad |x-y| \leqq a-x-y,$$

and consequently

$$K(x) \geqq K(a-x), \quad K(x-y) \geqq K(a-x-y).$$

Therefore $g'(x) \geqq 0$, with equality at $x = a/2$.

We now combine Lemmas 3 and 4 to prove Theorem 2. Let $f_0 = 1$, and define

$$\text{(9)} \qquad f_{n+1} = \int_0^a K(x-y)f_n(y)dy.$$

From Lemma 2 it follows that each $f_n(x)$ possesses properties 3a, b, and c, since f_0 does trivially. Lemma 3 tells us that

$$\phi(x) = \lim_{n\to\infty} f_n(x)/\lambda_M^n \tag{10}$$

is a characteristic function of $K(x-y)$ associated with λ_M, provided that it is not identically zero. That it is nontrivial follows from the fact that 1 as a positive function cannot be orthogonal to $f_M(x)$ which is also positive. It follows then that $f_M(x)$ possesses the stated properties, since there is only one characteristic function associated with λ_M.

The monotonicity property of $f_M(x)$ will play an important role in our second approximation technique. We shall not obtain as close a bound as before, however. Let us normalize our solution, which we know to be positive by the requirement $\int_0^a f(x)dx=1$. Integrating both sides of our integral equation between 0 and a we obtain

$$\begin{aligned}\lambda_M &= \int_0^a \left[\int_0^a K(x-y)dx\right] f(y)dy \\ &= 2\int_0^{a/2}\left[\int_0^y K(u)du + \int_0^{a-y} K(u)du\right] f(y)dy.\end{aligned} \tag{11}$$

From (11) we derive

$$\begin{aligned} 2c = \lambda_M &= 4c\int_0^{a/2} f(x)dx \\ &\quad - 2\int_0^{a/2}\left[\int_0^y K(u)du + \int_0^{a-y} K(u)du\right] f(y)dy \\ &= 2\int_0^{a/2}\left[c - \int_0^y K(u)du + c \right. \\ &\qquad\qquad \left. - \int_0^{a-y} K(u)du\right] f(y)dy \\ &= 2\int_0^{a/2}\int_y^\infty K(u)f(y)dudy \\ &\quad + 2\int_0^{a/2}\int_{a-y}^\infty K(u)f(y)dudy \geqq 0.\end{aligned} \tag{12}$$

Thus for Y in $(0, a/2)$,

$$
\begin{aligned}
|\lambda_M - 2c| = 2c - \lambda_M &\leqq 2\int_0^Y \int_y^\infty K(u)f(y)dudy \\
&\quad + 2\int_Y^{a/2}\int_y^\infty K(u)f(y)dudy \\
&\quad + \int_0^{a/2}\int_{a/2}^\infty K(u)f(y)dudy \\
&\leqq 2\int_0^Y\int_0^\infty K(u)f(y)dudy + 2\int_Y^{a/2}\int_Y^\infty K(u)f(y)dudy \\
&\quad + 2\int_0^{a/2} f(y)dy \int_{a/2}^\infty K(u)du \\
&\leqq 2c\int_0^Y f(y)dy + \int_0^{a/2}\int_Y^\infty K(u)f(y)dudy \\
&\quad + \int_{a/2}^\infty K(u)du \\
&= 2c\int_0^Y f(y)dy + \int_Y^\infty K(u)du + \int_{a/2}^\infty K(u)du.
\end{aligned} \tag{13}
$$

The original estimate of the authors involved $8c$ in place of $2c$ above. The simplification is due to the referee, whom we wish to thank for this and other helpful observations.

It remains to choose Y advantageously and estimate $\int_0^Y f(y)dy$. We have for $0 \leqq y \leqq a/2$, using the monotonic character of $f(x)$,

$$
\frac{1}{2}\int_0^{a/2} f(x)dx \geqq \int_y^{a/2} f(x)dx \geqq f(y)\left(\frac{a}{2} - y\right), \tag{14}
$$

and hence $f(y) \leqq 1/(a-2y)$. Therefore

$$
\int_0^Y f(y)dy \leqq Y/(a - 2Y). \tag{15}
$$

If $Y \to \infty$ in such a way that $Y/a \to 0$ as $a \to \infty$, we see that $\lambda_M \to 2c$. Choosing Y so that $2cY/(a-2Y) = \int_Y^\infty K(u)du$, we obtain a best possible error term from this procedure. For example, if $K(x) = e^{-x^2}$, we obtain as $a \to \infty$

$$
|\lambda_M - 2c| = O\left(\frac{(\log a)^{1/2}}{a}\right) \tag{16}
$$

which is inferior to the result stated in Theorem 1.

4. An approximation method for small *a*. Referring to (5) of §2, we see that it is possible to improve our estimates for λ_M by choosing, in place of $f=g=1$, functions which more nearly represent $f_M(x)$. Since we know the general form of $f_M(x)$ from Theorem 2, it would seem that two classes of functions which might yield good results are

$$f(x) = 1 + cx(a - x), \qquad c \geqq 0, \tag{1}$$

and

$$f(x) = \begin{cases} 1, & 0 \leqq x \leqq b < a/2, \\ c, & b \leqq x \leqq a - b, \\ 1, & a - b \leqq x \leqq a, \end{cases} \qquad c \geqq 1. \tag{2}$$

In each of these cases the numerical work connected with approximating to the largest characteristic root of the kernel $e^{-(x-y)^2}$ will not be overly complicated since all the integrals that occur may be evaluated in terms of known functions.

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