A NOTE ON NONCOMMUTATIVE POLYNOMIALS

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1. **Introduction.** We shall say that an integral domain[1] R satisfies condition (M) if any two nonzero elements of R have a nonzero common right multiple. In this note it is proved that if S is an extension of a ring R such that S is, roughly speaking, a noncommutative polynomial ring in one variable with R as a coefficient ring, and if R has the property (M), then S has property (M). In case R is a division ring, this result has been proved by Ore [**5**].[2] From our result it follows that the property (M) is preserved under an arbitrary number of extensions of the type described. It was first proved by Ore [**4**] that the condition (M) is necessary and sufficient in order that an integral domain R have a uniquely determined right quotient division ring. Our method is applied to prove that the Birkhoff-Witt algebra [**1**; **6**] of a solvable Lie algebra over an arbitrary field of characteristic zero satisfies condition (M), and consequently has a uniquely determined right quotient division ring.[3] It seems to be an unsolved problem to determine whether or not the Birkhoff-Witt algebra of an arbitrary Lie algebra satisfies condition (M).

2. **Ring extensions.** Let R and S be rings such that $R \subseteq S$. We shall say that S is an *extension of type* O of R if the following conditions are satisfied:

(a) R and S have the same identity element.

(b) S contains an element x not in R such that for each $r \in R$, $rx = xT(r) + D(r)$, $T(r), D(r) \in R$.

(c) R and x generate S, that is, upon applying (b) every element of S can be written in the form $\sum_{i=0}^{m} x^i r_i$, $r_i \in R$.

(d) $\sum_{i=0}^{m} x^i r_i = 0$ implies $r_i = 0$, $0 \leqq i \leqq m$.

From assumptions (b) and (d) it follows that $T(r+s) = T(r) + T(s)$, $T(rs) = T(r)T(s)$; $D(r+s) = D(r) + D(s)$ and $D(rs) = D(r)T(s) + rD(s)$. Ore proved [**5**, p. 481] that if R is an integral domain, and if we define the *degree* of an element $\sum_{i=0}^{m} x^i r_i$, $r_m \neq 0$, to be m, and deg 0 $= -\infty$, then the statement that $rx = xT(r) + D(r)$ is a necessary consequence of the assumption that

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[1] By an integral domain we mean a noncommutative associative ring with an identity element, containing no zero divisors.

[2] Numbers in brackets refer to the list of references at the end of the paper.

[3] I am indebted to Professor N. Jacobson for suggesting this problem.

$$\deg fg = \deg f + \deg g, \qquad f, g \in S. \tag{1}$$

Conversely, we observe that if S is an extension of type O of an integral domain R, then (1) holds for all f, $g \in S$.

LEMMA. *Let R be an integral domain satisfying condition* (M). *If S is an extension of R of type* O, *then S is an integral domain satisfying* (M).

PROOF.[4] That S is an integral domain follows from (1). Next we prove that S has a modified division process, namely, that if f and g are nonzero elements of S, then there exists a nonzero element $a \in R$, and elements h and $k \in S$ such that

$$fa = gh + k, \qquad \deg k < \deg g. \tag{2}$$

If $\deg f < \deg g$, then $f = g0 + f$, satisfying (2). If $\deg f \geqq \deg g$, then by induction we may assume that (2) holds for all elements of S having lower degree than f. Let $f = \sum_{i=0}^{n} x^i a_i$, $g = \sum_{i=0}^{m} x^i b_i$, where neither a_n nor b_m is equal to zero. Now $gx^{n-m} = x^n T^{n-m}(b_m) + \cdots$, where the positive integral powers of T are defined inductively by the formula $T^n(r) = T(T^{n-1}(r))$, $n = 2, 3, \cdots$. Since R satisfies (M), there exist nonzero elements c, $d \in R$ such that $a_n c = T^{n-m}(b_m)d$. If we set $f_1 = fc - gx^{n-m}d$, then the terms of highest degree cancel, and $\deg f_1 < \deg f$. Applying our induction hypothesis we have $f_1 e = gh + k$, $\deg k < m$, and

$$fce = g(x^{n-m}de + h) + k,$$

proving(2).

Now let f and g be nonzero elements of S; we wish to prove that they have a nonzero common right multiple. We observe first that by applying (2) it is clearly sufficient to consider the case where $\deg f < \deg g$. We prove by induction on the degree of g that g and any polynomial of degree less than $\deg g$ have a nonzero common right multiple. If $\deg g = 1$ and if $\deg r = 0$, that is, if $r \in R$, then by applying (2) we have $ga = rh + s$, $\deg s < 0$, hence $s = 0$, proving that g and r have a common right multiple. We assume now that if $\deg f < \deg g$, then f and any polynomial of lower degree have a nonzero common right multiple. Let $\deg f < \deg g$; then by (2) we have $gc = fp + q$, $\deg q < \deg f$. By our induction hypothesis f and q have

[4] The main idea in the proof of this lemma was suggested by Professor N. Jacobson. This argument replaces the author's much longer proof, which consisted in showing that S could be imbedded in a principal ideal domain, in which common multiples are known to exist, and then observing that as a consequence, S itself must satisfy (M).

a common right multiple, and hence f and g have a common right multiple as required.

THEOREM 1. *Let R be an integral domain satisfying the condition* (M). *Let S be a ring containing R which is the join of a well ordered sequence $\{S_\alpha\}$ of subrings, where α runs through some set of ordinal numbers. If α_0 is the least ordinal number in the set, let $R=S_{\alpha_0}$, and if $\alpha<\beta$, let $S_\alpha\subseteq S_\beta$. Let S_β be an extension of type* O *of S_α if $\beta=\alpha+1$, and an extension of type* O *of $\sum_{\alpha<\beta} S_\alpha$ if β is a limit ordinal. Then S is an integral domain satisfying condition* (M).

PROOF. We proceed by transfinite induction, assuming that S_α satisfies the conclusion of the theorem for all $\alpha<\beta$. It is sufficient to prove that S_β satisfies the conclusion of the theorem. If $\beta=\alpha+1$, then upon applying the lemma to S_α, we have our result. If β is a limit ordinal, then we apply the lemma to $\sum_{\alpha<\beta} S_\alpha$, first observing that $\sum_{\alpha<\beta} S_\alpha$ satisfies the hypothesis of the lemma, since it is the join of an increasing sequence of subrings, each of which satisfies these conditions. Therefore S_β satisfies the conclusions of the theorem for all β and since $S=\sum_\beta S_\beta$, S itself is an integral domain satisfying condition (M).

COROLLARY. *Let S satisfy the hypotheses of the theorem. Then S has a uniquely determined right quotient division ring.*

3. **An application.** We shall apply Theorem 1 to prove the following result.

THEOREM 2. *Let A be the Birkhoff-Witt algebra of a solvable Lie algebra L over an arbitrary field of characteristic zero. Then any two nonzero elements of A have a nonzero common right multiple.*

COROLLARY. *If A satisfies the hypothesis of Theorem* 2, *then A has a uniquely determined right quotient division ring.*[5]

PROOF OF THEOREM 2. The Birkhoff-Witt algebra is defined in [**1**; **3**], and in [**6**]. Harish-Chandra has proved [**3**, Corollary 1.2] that A is an integral domain. We shall prove that A satisfies the condition (M). First assume that the theorem has been proved when the base field is algebraically closed and of characteristic zero. Consider a solvable Lie algebra L over an arbitrary field Φ of characteristic zero, and let A be its Birkhoff-Witt algebra. Let Γ be the algebraic closure

[5] It is not difficult to prove, using the methods of [**2**, p. 150], that A can also be imbedded in a left quotient ring, and that the left and right quotient division rings are isomorphic.

of Φ, and let $\{\gamma_i\}$ be a basis for Γ over Φ. If we form the Kronecker product algebra $A\otimes\Gamma$ of A and Γ with respect to Φ, then $A\otimes\Gamma$ is the Birkhoff-Witt algebra of the solvable Kronecker product Lie algebra $L\otimes\Gamma$ over the algebraically closed field Γ, and we may assume that any two nonzero elements of $A\otimes\Gamma$ have a common right multiple. Observe that $A\subseteq A\otimes\Gamma$. If a and b are nonzero elements of A, then there exist nonzero elements a_1, $b_1\in A\otimes\Gamma$ such that $ab_1=ba_1$. We can write a_1 and b_1 uniquely in the form $a_1=\sum u_i\otimes\gamma_i$, $b_1=\sum w_j\otimes\gamma_j$, u_i and $w_j\in A$, and we have

$$\sum au_i\otimes\gamma_i=\sum bw_j\otimes\gamma_j.$$

Since these expressions are unique, for at least one index i, $au_i=bw_i\neq 0$. This proves that A satisfies the condition (M).

We may assume, therefore, that Φ is algebraically closed, and of characteristic zero. It follows from one of Lie's theorems that L has a basis $X_1,\cdots,X_n$ having the commutation rules

$$[X_i,X_j]=\sum_k \alpha_{ijk}X_k,\qquad \alpha_{ijk}\in\Phi,\tag{3}$$

where $\alpha_{ijk}=0$ if $k<\min(i,j)$. With respect to this basis we construct the Birkhoff-Witt algebra A of L, consisting of noncommutative polynomials in n letters $x_1,\cdots,x_n$ subject to the commutation rules

$$[x_i,x_j]=x_ix_j-x_jx_i=\sum_k \alpha_{ijk}x_k.\tag{4}$$

It is known [6] that the standard monomials $x_1^{e_1}x_2^{e_2}\cdots x_n^{e_n}$, $e_j\geqq 0$, form a basis for A over Φ. From (4) it follows that for each k, $1\leqq k<n$, the Φ-subspace A_k consisting of polynomials in $x_{n-k+1},\cdots,x_n$ alone is closed under multiplication and hence is a subring, and that for each k, the subring is sent into itself by the derivation $a\rightarrow ax_{n-k}-x_{n-k}a=[a,x_{n-k}]$. Furthermore $A_1\subseteq A_2\subseteq\cdots\subseteq A_n=A$. We prove next that for each $k<n$, A_{k+1} is an extension of type O of A_k. From what has already been said, it is sufficient to verify condition (d). Let

$$\sum_{i=0}^{r} x_{n-k}^{i}b_i=0,\qquad b_i\in A_k,\ 0\leqq i\leqq r.\tag{5}$$

We can write each b_i as a linear combination of the standard monomials in $x_{n-k+1},\cdots,x_n$ and the left side of (5) becomes a linear combination of standard monomials in $x_{n-k},\cdots,x_n$. Since the standard monomials are linearly independent, the coefficients of all the b_i must be zero, proving (d). Finally A_1, which is the polynomial algebra in x_n with coefficients in Φ, is a commutative integral domain, and consequently satisfies condition (M). Applying Theorem 1 to A, we conclude that A is an integral domain satisfying condition (M), and the proof of Theorem 2 is complete.

Added in proof, December 12, 1952. A proof of the fact that the Birkhoff-Witt algebra of an arbitrary Lie algebra satisfies condition (M) has been announced by D. Tamari (Bull. Amer. Math. Soc. Abstract 58-5-527.)

References

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