ABSOLUTE CONVERGENCE OF CONTINUED FRACTIONS[1]

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1. Introduction. Consider the continued fraction

$$f_1 + \cfrac{a_1}{b_1 - \cfrac{a_2}{b_2 - \cfrac{a_3}{b_3 - \cdots}}}, \tag{1.1}$$

where f_1 is a number, $\{a_1, a_2, a_3, \cdots\}$ is a sequence of nonzero numbers, and $\{b_1, b_2, b_3, \cdots\}$ is a sequence of numbers. We obtain conditions necessary and sufficient for (1.1) to converge absolutely, and we indicate their relationship to older sufficient conditions. We find a new characterization of positive definite continued fractions, whose importance is emphasized by the fact (Theorem 4.2) that if (1.1) converges, then there is a positive definite continued fraction which is a contraction of (1.1). We also obtain new sufficient conditions for absolute convergence of positive definite continued fractions.

2. Continued fractions and sequences of linear fractional transformations. In this paper, a subscript p denotes a positive integer. By the generator of (1.1) we mean the sequence $\{t_1(u), t_2(u), t_3(u), \cdots\}$ of linear fractional transformations such that $t_1(u) = f_1 + a_1/(b_1 - u)$ and $t_{p+1}(u) = t_p[a_{p+1}/(b_{p+1} - u)]$ for $p \geqq 1$. We denote this sequence by $t(u)$.

REMARK 2.1. For a sequence $s(u)$ of linear fractional transformations to be the generator of a continued fraction, it is necessary and sufficient that $s_1(\infty) \neq \infty$ and $s_p(0) = s_{p+1}(\infty)$ for $p \geqq 1$.

By the sequence of approximants of (1.1) we mean the sequence $\{f_1, f_2, f_3, \cdots\}$ such that $f_p = t_p(\infty)$ for $p \geqq 1$. We denote this sequence by f.

REMARK 2.2. For a sequence x of points in the complex plane to be the sequence of approximants of a continued fraction with nonzero partial numerators, it is necessary and sufficient that $x_1 \neq \infty$ and $x_p \neq x_{p+1}$ for $p \geqq 1$.

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[1] This paper is based on a thesis written under the direction of Professor H. S. Wall of the Department of Pure Mathematics, The University of Texas.

If f has the property that there exists a positive integer n such that (1) the sequence $\{f_n, f_{n+1}, f_{n+2}, \cdots\}$ is bounded and (2) either $n=1$ or $f_{n-1}=\infty$, then by B_f we mean the set of all sequences R such that for $p \geqq 1$

(2.1)
(i) R_p is a circle plus its interior,
(ii) $R_p \supset R_{p+1}$, and
(iii) f_p is in R_p if $p \geqq n$.

THEOREM 2.1. *If f is bounded, then for R to be a member of B_f it is necessary and sufficient that*

(2.2)
(i) *R_1 is a circle plus its interior,*
(ii) *if $p \geqq 1$, then $t_p^{-1}(R_p)$ is a closed half-plane or a circle plus its exterior, and*
(iii) *if $p \geqq 1$, then $t_p^{-1}(R_p) \supset t_p^{-1}(R_{p+1})$.*

Moreover, if R is a sequence in B_f, and if $p \geqq 1$, then $t_p^{-1}(R_p)$ is a closed half-plane if f_p is a boundary point of R_p, or is a circle plus its exterior if f_p is an interior point of R_p.

PROOF. The theorem is a direct consequence of the definitions of f, $t(u)$, and B_f.

We denote by h the sequence $\{h_1, h_2, h_3, \cdots\}$ of points in the complex plane such that if $p \geqq 1$ then $h_p = t_p^{-1}(\infty)$. From the relations $t_1(b_1) = \infty$ and $t_{p+1}(u) = t_p[a_{p+1}/(b_{p+1}-u)]$, it follows that

$$h_1 = b_1 \quad \text{and} \quad h_{p+1} = b_{p+1} - a_{p+1}/h_p \qquad \text{for } p \geqq 1. \tag{2.3}$$

If $p \geqq 1$, then $t_p(\infty) = f_p$, $t_p(0) = f_{p+1}$, and $t_{p+1}(b_{p+1}) = f_p$; so that

$$\begin{aligned} f_p &= \infty \text{ if and only if } h_p = \infty, \\ f_{p+1} &= \infty \text{ if and only if } h_p = 0, \text{ and} \\ f_p &= \infty \text{ if and only if } h_{p+1} = b_{p+1}. \end{aligned} \tag{2.4}$$

If $p \geqq 1$, and if $f_p \neq \infty$ and $f_{p+1} \neq \infty$, then

$$t_p(u) = f_p + \frac{h_p(f_{p+1} - f_p)}{h_p - u}. \tag{2.5}$$

If $p \geqq 1$, and if $f_p \neq \infty$, $f_{p+1} \neq \infty$, and $f_{p+2} \neq \infty$, then

$$\frac{f_{p+1} - f_{p+2}}{f_p - f_{p+1}} = \frac{b_{p+1} - h_{p+1}}{h_{p+1}} = \frac{a_{p+1}}{h_p h_{p+1}}. \tag{2.6}$$

3. Conditions necessary and sufficient for absolute convergence. If x is a sequence of points in the complex plane, the statement that x converges absolutely means that there exists a positive integer n

such that (1) if $p \geqq n$, then $x_p \neq \infty$ and (2) $\sum_{p=n}^{\infty} |x_p - x_{p+1}|$ converges. The statement that a continued fraction converges absolutely means that its sequence of approximants converges absolutely.

THEOREM 3.1. *For* (1.1) *to converge absolutely, it is necessary and sufficient that there exist a positive integer* n, *a sequence* s *of numbers, and a sequence* q *of numbers such that*

(3.1)
(i) $s_p > 0$ *and* $q_p \neq 0$ *for* $p \geqq n$, *and* $\sum_{p=n}^{\infty} s_p$ *converges,*
(ii) *there is a sequence* R *in* B_f *such that if* $p \geqq n$, *then* $t_p^{-1}(R_p)$ *is the region defined by the inequality* $s_p|u| \leqq |u - q_p|$, *and*
(iii) *there is a sequence* R' *in* B_f *such that if* $p \geqq n$, *then* q_p *is in* $t_p^{-1}(R_p')$.

PROOF. A. Suppose that there exist such an integer n and such sequences s and q. Let m denote an integer such that if $p \geqq m$, then $p \geqq n$ and f_p is in R_p. Now R_p is a circle plus its interior, ∞ is not in R_p, and $h_p = t_p^{-1}(\infty)$ is not in $t_p^{-1}(R_p)$; hence if $p \geqq m$, then $s_p|h_p| > |h_p - q_p|$, or $s_p > |(h_p - q_p)/h_p|$. Moreover, if $p \geqq m$, then by (2.4) and (2.5),

$$|(f_{p+1} - f_p)/[t_p(q_p) - f_p]| = |(h_p - q_p)/h_p| < s_p.$$

By hypothesis, f_p is in R_m and $t_p(q_p)$ is in R_m', and consequently there exists a number M such that if $p \geqq m$, then $t_p(q_p) - f_p < M$, so that $|f_{p+1} - f_p| < Ms_p$. Since $\sum_{p=n}^{\infty} s_p$ converges, (1.1) converges absolutely.

B. Suppose that (1.1) converges absolutely. Let n denote the positive integer such that if $p \geqq n$, then $f_p \neq \infty$ and such that either $n = 1$ or $f_{n-1} = \infty$. Let R_n be a circle plus its interior, with radius r and center c such that if $p \geqq n$, then $3r > 4|f_p - c| > 2r$. Let R_n' be a circle plus its interior with radius r' and center c, such that $R_n' \supset R_n$ and such that if $p \geqq n$, then the inversion of f_p in the boundary of R_n is in R_n'. For $p \geqq 1$, let $R_p = R_n$ and $R_p' = R_n'$. Then R is in B_f and R' is in B_f.

For $p \geqq n$, let $t_p(q_p)$ be the inversion of f_{p+1} in the boundary of R_p. By construction, $t_p(q_p)$ is in R_p', so that q_p is in $t_p^{-1}(R_p')$. Moreover, if $p \geqq n$, then there exists a positive number s_p' such that R_p is the region defined by $s_p'|u - f_{p+1}| \leqq |u - t_p(q_p)|$; and since $3r > 4|f_{p+1} - c| > 2r$, there exist positive numbers D and s' such that $|t_p(q_p) - f_p| \geqq D$ and $s_p' \leqq s'$ for $p \geqq n$. By (2.5), $t_p^{-1}(R_p)$ is the region defined by $s_p|u| \leqq |u - q_p|$, where $s_p = s_p'|(f_{p+1} - f_p)/[t_p(q_p) - f_p]| < |f_{p+1} - f_p| s'/D$. Hence $\sum_{p=n}^{\infty} s_p$ converges. This completes the proof.

LEMMA 3.2a. *If* s *is a sequence of positive numbers, then for* $\sum_{p=1}^{\infty} s_p$ *to converge it is necessary and sufficient that there exist a sequence* d *of* *positive numbers such that for* $p \geqq 1$

$$\frac{s_{p+1}}{s_p} \leqq \frac{d_p}{1 + d_{p+1}}.$$

PROOF. Suppose that d is such a sequence. If $p \geqq 1$, then $s_{p+1} + s_{p+1}d_{p+1} \leqq s_p d_p$; and by induction, if n is an integer greater than p, then $\sum_{k=p+1}^{n} s_k + s_n d_n \leqq s_p d_p$, so that $\sum_{k=1}^{n} s_k < \sum_{k=1}^{p} s_k + s_p d_p$. Hence $\sum_{k=1}^{\infty} s_k$ converges.

Suppose that $\sum_{k=1}^{\infty} s_k$ converges. Let r be a sequence of non-negative real numbers such that $\sum_{k=1}^{\infty} r_k$ converges, and for $p \geqq 1$, let d_p be the positive number such that $s_p d_p = \sum_{k=p+1}^{\infty} (r_k + s_k)$. Then $s_p d_p = r_{p+1} + s_{p+1} + s_{p+1} d_{p+1} \geqq s_{p+1} + s_{p+1} d_{p+1}$, so that $s_{p+1}/s_p \leqq d_p/(1 + d_{p+1})$. This completes the proof.

REMARK 3.1. From the above proof it follows that if in Lemma 3.2a the statement $s_{p+1}/s_p \leqq d_p/(1 + d_{p+1})$ is replaced by either of the statements

$$\frac{s_{p+1}}{s_p} < \frac{d_p}{1 + d_{p+1}}, \qquad \frac{s_{p+1}}{s_p} = \frac{d_p}{1 + d_{p+1}},$$

then the resulting lemma is true.

EXAMPLE 3.1. Let $a > -1$, $b > a + 1$, and $d_p = (a + p)/(b - a - 1)$ for $p \geqq 1$. By Lemma 3.2a, the series

$$1 + \frac{a + 1}{b + 1} + \frac{(a + 1)(a + 2)}{(b + 1)(b + 2)} + \cdots$$

converges.

LEMMA 3.2b. *For f to converge absolutely, it is necessary and sufficient that there exist a positive integer n and a sequence d of positive numbers such that, for* $p \geqq n$,

(i) $d_p > 1 + d_{p+1}$ *if* $f_{p+1} = \infty$ *or if* $f_p = f_{p+2} = \infty$ *and*

(ii) $d_p |f_p - f_{p+1}| > (1 + d_{p+1}) |f_{p+1} - f_{p+2}|$ *if*

(3.2) (a) $f_{p+1} \neq \infty$ *and*

(b) $f_p \neq \infty$ *or* $f_{p+2} \neq \infty$.

PROOF. If f converges absolutely, then there exists a positive integer n such that $f_p \neq \infty$ if $p \geqq n$; and by Remark 3.1 there exists a sequence d of positive numbers such that (ii) holds for $p \geqq n$.

Suppose that there exist a positive integer n and a sequence d of positive numbers such that (3.2) holds for $p \geqq n$. We first show that if $p \geqq n + d_n$, then $f_p \neq \infty$. Suppose that m is an integer, that $m \geqq n + d_n$, and that $f_m = \infty$. Then for $p = m - 1$, the relation (i) holds by hypothesis, and $d_{m-1} > 1 + d_m > 1$. Since $f_m = \infty$, it follows (Remark

2.2) that $f_{m-1}\neq\infty$. If $f_{m-2}\neq\infty$, therefore, (ii) must hold for $p=m-2$; but this is impossible, since $f_m=\infty$. Hence $f_{m-2}=\infty$, and (i) holds for $p=m-2$, so that $d_{m-2}>1+d_{m-1}>2$. If $m>n+2$, then (i) must hold for $p=m-3$, and $d_{m-3}>3$. If $m>n+3$, then $f_{m-4}=\infty$ and $d_{m-4}>4$. By induction, $d_n>m-n$, so that $m<n+d_n$. Hence the assumption that $f_m=\infty$ is false; and if $p\geqq n+d_n$, then $f_p\neq\infty$. By Lemma 3.2a, f converges absolutely. This completes the proof.

THEOREM 3.2. *For* (1.1) *to converge absolutely, it is necessary and sufficient that there exist a positive integer n and a sequence d of positive numbers such that, for* $p\geqq n$,

(3.3)
(i) *if* $b_{p+1}=0$, *then* $d_p>1+d_{p+1}$, *and*
(ii) *if* $b_{p+1}\neq 0$ *and if* $t_{p+1}^{-1}(K_{p+1})$ *is the region defined by* $d_p|u| \leqq(1+d_{p+1})|u-b_{p+1}|$, *then* K_{p+1} *is a circle plus its interior.*

PROOF. The conditions (3.2) of Lemma 3.2b can be written

(a) $d_p>1+d_{p+1}$ if $f_p=f_{p+2}$,
(b) $d_p>1+d_{p+1}$ if $f_p\neq f_{p+2}$ and $f_{p+1}=\infty$,
(c) $d_p|f_p-f_{p+1}|>(1+d_{p+1})|f_{p+1}-f_{p+2}|$ if $f_p\neq f_{p+2}$ and $f_{p+1}\neq\infty$.

Since $f_p=t_{p+1}(b_{p+1})$, $f_{p+2}=t_{p+1}(0)$, and $\infty=t_{p+1}(h_{p+1})$, the first two of these conditions can be written

(a′) $d_p>1+d_{p+1}$ if $b_{p+1}=0$,
(b′) $d_p>1+d_{p+1}$ if $b_{p+1}\neq 0$ and $h_{p+1}=\infty$;

as for the third, where $b_{p+1}\neq 0$ and $h_{p+1}\neq\infty$, similar consideration of the two cases (1) $h_{p+1}=0$ and (2) $h_{p+1}=b_{p+1}$, and use of (2.6) for the case (3) $h_{p+1}\neq 0$, $h_{p+1}\neq b_{p+1}$, shows that (c) may be written

(c′) $d_p|h_{p+1}|>(1+d_{p+1})|h_{p+1}-b_{p+1}|$ if $b_{p+1}\neq 0$ and $h_{p+1}\neq\infty$.

Now if $t_{p+1}^{-1}(K_{p+1})$ is defined by $d_p|u|\leqq(1+d_{p+1})|u-b_{p+1}|$, where $d_p>0$, $d_{p+1}>0$, and $b_{p+1}\neq 0$, then for K_{p+1} to be a circle plus its interior, it is necessary and sufficient that the point $h_{p+1}=t_{p+1}^{-1}(\infty)$ be exterior to $t_{p+1}^{-1}(K_{p+1})$. Hence the conditions (a′), (b′), and (c′) are equivalent to (3.3), and the theorem now follows from Lemma 3.2b. This completes the proof.

REMARK 3.2. If $a_1=1$ and $b_p=1$ and $a_{p+1}=-c_p$ for $p\geqq 1$, then (1.1) is the continued fraction

$$\frac{1}{1+\frac{c_1}{1+\frac{c_2}{1+\cdots}}}, \tag{3.4}$$

where c is a sequence of nonzero numbers. If, in the notation of Theorem 3.2, $r_p = d_p/(1+d_{p+1})$, then $t_{p+1}^{-1}(K_{p+1})$ is defined by the inequality $r_p|u| \leqq |u-1|$. The condition $t_p^{-1}(K_p) \supset t_p^{-1}(K_{p+1})$ gives the inequalities (5.5), p. 376, of Lane and Wall [1][2] for $p \geqq 1$. The condition $t_p^{-1}(K_{p-1}) \supset t_p^{-1}(K_{p-1p+1})$ gives, for $p \geqq 2$, the inequalities (13.) of Scott and Wall [2].

4. A characterization of positive definite continued fractions. The continued fraction (1.1) is said to be positive definite[3] if

(4.1)
(i) $I(b_1) > 0$ and $I(b_p) \geqq 0$ for $p > 1$, and
(ii) there exists a sequence g of numbers such that $0 < g_1 \leqq 1$ and, for $p \geqq 1$, $0 \leqq g_{p+1} \leqq 1$ and

$$|a_{p+1}| - R(a_{p+1}) \leqq 2I(b_p)I(b_{p+1})(1-g_p)g_{p+1}.$$

If F is a continued fraction, the statement that F is equivalent to (1.1) means that the sequence of approximants of F is the sequence of approximants of (1.1).

REMARK 4.1. If F is a continued fraction, and if $t'(u)$ is the generator of F, then for F to be equivalent to (1.1) it is necessary and sufficient that there exist a sequence σ of nonzero numbers such that $t_p'(u) = t_p(u/\sigma_p)$ for $p \geqq 1$. If σ is such a sequence, then F is the continued fraction

$$f_1 + \cfrac{\sigma_1 a_1}{\sigma_1 b_1 - \cfrac{\sigma_1\sigma_2 a_2}{\sigma_2 b_2 - \cfrac{\sigma_2\sigma_3 a_3}{\sigma_3 b_3 - \cdots}}}.$$

THEOREM 4.1. *For* (1.1) *to be equivalent to a positive definite continued fraction, it is necessary and sufficient that there exist a sequence R in B_f such that if $p \geqq 1$, then $t_p^{-1}(R_p)$ is a closed half-plane; i.e., if $p \geqq 1$, then f_p is a boundary point of R_p.*

PROOF. Let $t^{-1}(R)$ be a sequence of closed half-planes. Then there exist a sequence σ of nonzero numbers and a sequence k of real numbers such that if $p \geqq 1$, then $t_p^{-1}(R_p)$ is defined by $R(\sigma_p u) \leqq k_p$. We show first that for R to be in B_f it is necessary and sufficient that

(4.2)
(i) $0 \leqq k_1 < R(\sigma_1 b_1)$ and $0 \leqq k_p \leqq R(\sigma_p b_p)$ for $p > 1$, and
(ii) $R(\sigma_p\sigma_{p+1}a_{p+1}) + |\sigma_p\sigma_{p+1}a_{p+1}| \leqq 2k_p R(\sigma_{p+1}b_{p+1} - k_{p+1})$ for $p \geqq 1$.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

[3] This is an adaptation to (1.1) of the definition on pp. 67–71 of [3], where it is assumed that $g_1 I(b_1) > 0$; e.g., in formula (17.3) of [3].

For R_1 to be a circle plus its interior, it is necessary and sufficient that the point $t_1^{-1}(\infty)=b_1$ be exterior to $t_1^{-1}(R_1)$; i.e., that $R(\sigma_1 b_1)>k_1$. If $p\geqq 1$, then ∞ is a boundary point of $t_p^{-1}(R_p)$, and $f_p=t_p(\infty)$ is a boundary point of R_p; similarly, f_{p+1} is a boundary point of R_{p+1}. If $R_p\supset R_{p+1}$, then the point $t_p^{-1}(f_{p+1})=0$ is in $t_p^{-1}(R_p)$, or $0\leqq k_p$; moreover, the point $t_{p+1}^{-1}(f_p)=b_{p+1}$ is not an interior point of $t_{p+1}^{-1}(R_{p+1})$, or $R(\sigma_{p+1}b_{p+1})\geqq k_{p+1}$. Hence for $t^{-1}(R)$ to be in B_f, the conditions (i) of (4.2) are necessary.

Suppose that (i) of (4.2) holds. Then for $p\geqq 1$, $t_p^{-1}(R_{p+1})$ is defined by

$$R(\bar\sigma_{p+1}\bar a_{p+1}u) \geqq 0 \quad \text{if} \quad R(\sigma_{p+1}b_{p+1}) = k_{p+1}, \quad \text{or}$$

$$\left| u - \frac{\sigma_{p+1}a_{p+1}}{2R(\sigma_{p+1}b_{p+1} - k_{p+1})}\right| \leqq \frac{|\sigma_{p+1}a_{p+1}|}{2R(\sigma_{p+1}b_{p+1} - k_{p+1})} \tag{4.3}$$

$$\text{if} \quad R(\sigma_{p+1}b_{p+1}) > k_{p+1}.$$

Hence if (i) of (4.2) holds, then (ii) is a condition necessary and sufficient for the relations $R_p\supset R_{p+1}$ to hold for $p\geqq 1$. We conclude that $t^{-1}(R)$ is in B_f if and only if (4.2) holds.

If for $p\geqq 1$ we take $\sigma_p=-i$ and $k_p=(1-g_p)R(-ib_p)$, where $g_p=1$ if $k_p=0$, the theorem now follows from (4.1) and Remark 4.1. This completes the proof.

REMARK 4.2. By Theorem 4.1, a bounded increasing infinite sequence of real numbers is the sequence of approximants of a positive definite continued fraction. More generally, if x is a sequence of numbers, if $x_p\neq x_{p+1}$ for $p\geqq 1$, and if there exists a number c such that $|x_p-c|\geqq|x_{p+1}-c|$ for $p\geqq 1$, then x is the sequence of approximants of a positive definite continued fraction.

THEOREM 4.2. *If* (1.1) *converges, then there exists a positive definite continued fraction whose sequence of approximants is a subsequence of* f.

PROOF. Let c be the number such that $f_p\rightarrow c$ as $p\rightarrow\infty$. Then there exists an infinite subsequence, x, of f such that if $p\geqq 1$, then $x_p\neq\infty$ and $|x_p-c|>|x_{p+1}-c|$. By Remark 4.2, x is the sequence of approximants of a positive definite continued fraction. This completes the proof.

5. Absolute convergence of positive definite continued fractions.

Throughout this section we suppose that (1.1) is equivalent to a positive definite continued fraction, and that k is a sequence of real numbers such that R is in B_f, where, for $p\geqq 1$, $t_p^{-1}(R_p)$ is the closed half-plane $R(u)\leqq k_p$. The conditions (4.2) hold, therefore, with $\sigma_p=1$, and $t_p^{-1}(R_{p+1})$ is the region defined by (4.3), for $p\geqq 1$.

THEOREM 5.1. *If there exist a positive integer n and a positive number M such that* $|a_{p+1}| \leqq Mk_pR(b_{p+1}-k_{p+1})$ *for* $p \geqq n$, *then* (1.1) *converges absolutely.*

PROOF. Since, by hypothesis, (1.1) is equivalent to a positive definite continued fraction, its sequence of approximants is bounded; and by (2.4), $h_p \neq \infty$ and $h_p \neq 0$ for $p \geqq 1$. Moreover, if $p \geqq n$, then $t_p^{-1}(R_{p+1})$ is a circle plus its interior; let v_p be the point of $t_p^{-1}(R_{p+1})$ farthest from h_p. By (2.5),

$$\left|\frac{f_{p+1} - f_p}{t_p(v_p) - f_p}\right| = \left|\frac{h_p - v_p}{h_p}\right| \leqq 1 + \left|\frac{v_p}{h_p}\right|.$$

Since the origin is a boundary point of $t_p^{-1}(R_{p+1})$, $|v_p|$ is less than or equal to the diameter of $t_p^{-1}(R_{p+1})$, or $|v_p| \leqq |a_{p+1}|/R(b_{p+1}-k_{p+1})$; hence $|v_p| < Mk_p$. Since R_p is a circle plus its interior, h_p is not in the closed half-plane $R(u) \leqq k_p$; so $|h_p| \geqq R(h_p) > k_p$. Finally, by (2.5), $t_p(v_p)$ is the point of R_{p+1} nearest f_p; so $|t_p(v_p) - f_p| \leqq 2(r_p - r_{p+1})$, where for $p \geqq 1$, r_p is the radius of R_p. We now conclude that if $p \geqq n$, then $|f_{p+1} - f_p| < 2(1+M)(r_p - r_{p+1})$. Since $\sum_{p=n}^{\infty} (r_p - r_{p+1})$ is a convergent positive-term series, (1.1) converges absolutely. This completes the proof.

COROLLARY 5.1a. *If there exist a sequence g and a positive number M such that, for* $p \geqq 1$,

(i) $0 < g_p < 1$,
(ii) $|c_p| - R(c_p) \leqq 2(1-g_p)g_{p+1}$, *and*
(iii) $|c_p| < M(1-g_p)g_{p+1}$,

then the continued fraction (3.4) *converges absolutely.*

REMARK 5.1. The above corollary is a true generalization of the convergence condition $|c_p| \leqq (1-g_p)g_{p+1}$, $p \geqq 1$, of Pringsheim [**4**]; compare it with the condition $|c_p| - R(c_p) \leqq 2r(1-g_p)g_{p+1}$, where $0 < r < 1$, $p \geqq 1$, on pp. 142–143 of [**3**].

REMARK 5.2. It should be noted that in Theorem 5.1 and its corollary we do not conclude that the common part of $R_1, R_2, R_3, \cdots$ is a point. Actually there exists an absolutely convergent positive definite continued fraction which has the property that if $t^{-1}(R)$ is a sequence of closed half-planes such that R is in B_f, then the common part of $R_1, R_2, R_3, \cdots$ is a circle plus its interior. We give the following example. Let s be a decreasing sequence of positive numbers such that $\sum_{p=1}^{\infty} s_p$ converges. For $p \geqq 1$, let each of R_{3p-2}, R_{3p-1}, and R_{3p} be the region defined by $|u-(s_p-1)| \leqq 1+s_p$, and let f_{3p-2}, f_{3p-1}, and f_{3p} be boundary points of R_{3p} such that $\arg f_{3p-2}=0$, $\arg f_{3p-1}=1$, and $\arg f_{3p}=-1$. Then f is the sequence of approximants of an absolutely convergent positive definite continued fraction. If R' is a sequence in B_f such that f_p is a boundary point of R_p' for $p\geqq 1$, then $R'_{3p-2}\supset R_{3p-2}$ for $p\geqq 1$, and hence the common part of $R_1', R_2', R_3', \cdots$ is a circle plus its interior.

THEOREM 5.2. *Let* $e_p=|a_{p+1}|/[2k_pR(b_{p+1}-k_{p+1})-R(a_{p+1})]$ *for* $p\geqq 1$. *If* $\sum_{p=1}^{\infty}(1-e_p)$ *diverges, then* (1.1) *converges. If there exists a sequence* d *of positive numbers such that* $e_p(2+2d_{p+1}-d_p)\leqq d_p$ *for* $p\geqq 1$, *then* (1.1) *converges absolutely.*

PROOF. A. We show first that $r_{p+1}/r_p\leqq 2e_p/(1+e_p)$ for $p\geqq 1$, where r_p is the radius of R_p. If $R(b_{p+1})=k_{p+1}$, then by (4.2) $a_{p+1}<0$ and hence $e_p=1$, so that the relation $r_{p+1}/r_p\leqq 2e_p/(1+e_p)$ holds. If $R(b_{p+1})>k_{p+1}$, then $t_p^{-1}(R_{p+1})$ is a circle plus its interior; let v_p be the point of $t_p^{-1}(R_{p+1})$ farthest from h_p, and let w_p be the point of $t_p^{-1}(R_{p+1})$ nearest h_p. By (2.5), $t_p(v_p)$ is the point of R_{p+1} nearest f_p, and $t_p(w_p)$ is the point of R_{p+1} farthest from f_p; so $2r_{p+1}=|t_p(v_p)-t_p(w_p)|$ and $2r_p\geqq|t_p(w_p)-f_p|$. But by (2.5), $[t_p(v_p)-t_p(w_p)]/[t_p(w_p)-f_p]=|(v_p-w_p)/(h_p-v_p)|$; hence $r_{p+1}/r_p\leqq|(v_p-w_p)/(h_p-v_p)|$. Since the diameter of $t_p^{-1}(R_{p+1})$ is $|v_p-w_p|=|a_{p+1}|/R(b_{p+1}-k_{p+1})$, and since the distance from h_p to v_p is $|h_p-v_p|>k_p+[|a_{p+1}|-R(a_{p+1})]/2R(b_{p+1}-k_{p+1})$, it follows that

$$\frac{r_{p+1}}{r_p}\leqq\frac{2|a_{p+1}|}{2k_pR(b_{p+1}-k_{p+1})-R(a_{p+1})+|a_{p+1}|}=\frac{2e_p}{1+e_p}.$$

B. Suppose that $\sum_{p=1}^{\infty}(1-e_p)$ diverges. Now by definition and by (4.2), $0<e_p\leqq 1$; so $\sum_{p=1}^{\infty}(1-e_p)/(1+e_p)$ diverges. But $1-r_{p+1}/r_p\geqq 1-2e_p/(1+e_p)=(1-e_p)/(1+e_p)$. Hence if for $p\geqq 1$, $s_p=1-r_{p+1}/r_p$, then $\sum_{p=1}^{\infty}s_p$ is a divergent series whose terms are nonnegative real numbers. Since $r_{p+1}=r_1(1-s_1)(1-s_2)\cdots(1-s_p)$, it follows that $r_p\to 0$ as $p\to\infty$, and consequently (1.1) converges.

C. Suppose that there exists a sequence d of positive numbers such that $e_p(2+2d_{p+1}-d_p)\leqq d_p$ for $p\geqq 1$. Then for $p\geqq 1$, $r_{p+1}/r_p\leqq 2e_p/(1+e_p)\leqq d_p/(1+d_{p+1})$; and by Lemma 3.2a, $\sum_{p+1}^{\infty}r_p$ converges. Since $|f_{p+1}-f_p|\leqq 2r_p$, (1.1) converges absolutely. This completes the proof of the theorem.

EXAMPLE 5.1. Let s be a positive number greater than 4. If $0<c_p\leqq p/s$ for $p\geqq 1$, then the continued fraction (3.4) converges absolutely. This can be seen by taking $k_p=1/2$ and $d_p=4p/(s-4)$ in Theorem 5.2. In Corollary 6.1a, p. 380, of [1], it was required that an infinite subsequence of c be bounded.

REMARK 5.3. *If* $c_p = p(p+x)/(1+x)^2$ for $p \geqq 1$, it can be shown that (3.4) converges absolutely for $x > 0$; it was shown on p. 379 of [1] that if $x = 0$, then (3.4) converges but does not converge absolutely.

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