THE IDENTITIES OF PI-RINGS

S. A. AMITSUR

1. **Introduction.** Let S be a ring with a ring of operators Ω. The ring S is said to be a PI-ring (that is, a ring with a polynomial identity) over Ω if S satisfies a polynomial identity $g(x_1, \cdots, x_n)=0$ with coefficients in Ω. We make correspond to every PI-ring S a universal PI-ring which satisfies the identities and only the identities of S. The universal ring is maximal in the sense that the rings satisfying the identities of S are characterized as the homomorphic images of the universal ring. The Jacobson radical [3] of the universal ring is shown to be its maximal nil ideal. This fact and the study of the universal rings which correspond to total matrix algebras yield a representation of the free ring over Ω, which in turn implies that every ring (not necessarily a PI-ring) is a homomorphic image of a subdirect sum of total matrix rings of finite order over the ring of all integers. In particular, if the ring considered is nilpotent, the orders of the matrix rings are bounded. Another result obtained by studying the radical of the universal rings is that every PI-ring satisfies an identity of the form: $S_{2n}^m(x)=0$ where $S_{2n}(x)$ is the standard polynomial of degree $2n$ (see [2]).

2. **Universal PI-rings.** In what follows we assume that Ω is an integral domain which contains an infinite number of elements and such that: (1) $\alpha S=0$, $\alpha\in\Omega$, implies $\alpha=0$. (2) $\alpha(rs)=(\alpha r)s=r(\alpha s)$ for every $\alpha\in\Omega$ and $r, s\in S$. Let $\{x\}$ be an infinite set of indeterminates over Ω. We denote by $\Omega[x]$ the free ring generated by the set $\{x\}$ and Ω. Following Specht [1, p. 565] we call an ideal Q in $\Omega[x]$ a *T-ideal* if (1) $\alpha p(x)\in Q$, $\alpha\in\Omega$, $\alpha\not\equiv 0$, implies $p(x)\in Q$, and (2) $Q^T\subseteq Q$ for every homomorphism T: $x\rightarrow t(x)$ of $\Omega[x]$ onto its subrings.

If S is a PI-ring over Ω, the totality of the identities of S constitute a nonzero T-ideal Q_S in $\Omega[x]$. We refer to this ideal as the *ideal of identities* of S and the quotient ring $\Omega[x]/Q_S$ will be called the *universal ring* of S.

It was shown in [1] that in the case of PI-rings of characteristic zero, which is the case where Ω is the integral domain of all integers, the converse also holds. That is: if Q is a nonzero T-ideal, then $\Omega[x]/Q$ is a PI-ring whose ideal of identities is the ideal Q. The following lemma is a generalization of this result:

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LEMMA 1. *Let P be an ideal in $\Omega[x]$.*[1] *The quotient $\Omega[x]/P$ is a* PI-*ring if and only if P contains a nonzero T-ideal; and if this condition holds, the ideal of identities of $\Omega[x]/P$ is the maximal T-ideal contained in P.*

PROOF. The set of all polynomials $g(x)\in P$ such that $g(x)^T\in P$ for every homomorphism T constitute the maximal T-ideal Q contained in P. If $Q\neq 0$, let $g(x_1, \cdots, x_n)$ be any nonzero polynomial of Q. Since Q is a T-ideal, for any set of polynomials $t_1(x), \cdots, t_n(x)$, $g(t_1(x), \cdots, t_n(x))\in Q$. Thus if $'$ denotes reduction modulo P, it follows by $P\supseteq Q$ that:

$$g'(t_1(x), \cdots, t_n(x)) = g(t_1'(x), \cdots, t_n'(x)) = 0.$$

This proves that $g(x_1, \cdots, x_n)=0$ is an identity satisfied by $\Omega[x]/P$. Conversely, if $g(x_1, \cdots, x_n)=0$ is satisfied by $\Omega[x]/P$, for any set of polynomials $t_1(x), \cdots, t_n(x)$ we have $g(t_1'(x), \cdots, t_n'(x))=0$. This means that $g(t_1(x), \cdots, t_n(x))\in P$, which proves that the T-ideal generated by $g(x)$ is a T-ideal contained in Q. Hence $g(x)\in Q$ and the proof is completed.

Note that if the set $\{x\}$ is sufficiently large, the ring S is a homomorphic image of its universal ring. Let R be a PI-ring which satisfies the identities of S; then $Q_R\supseteq Q_S$, where Q_R and Q_S are the ideals of identities of R and S respectively. Hence the universal ring of R is a homomorphic image of the universal ring of S. In particular, if the cardinal number of the set $\{x\}$ is not smaller than the cardinal numbers of R and S, the ring R is also a homomorphic image of the universal ring of S. On the other hand, by Lemma 1 the universal ring of S satisfies the identities of S; it follows, therefore, that every homomorphic image of the universal ring of S satisfies the identities of S. Thus we have proved:

THEOREM 1. *A ring R satisfies all identities of S if and only if R is a homomorphic image of the universal ring of S.*

It should be remarked that if the set $\{x\}$ is fixed, the preceding theorem holds only for rings R with cardinal number not greater than that of the set $\{x\}$; but if R is given, one can choose $\{x\}$ so that the preceding theorem will be true.

In what follows "radical" and "semisimplicity" will be used in the sense of Jacobson [**3**].

Let Q be a nonzero T-ideal.

LEMMA 2. *Let $p(x_1, \cdots, x_n)\in Q$ and let*

[1] We assume that from $\alpha\neq 0$, $\alpha\in\Omega$, $\alpha p(x)\in P$ it follows that $p(x)\in P$.

$$p(x_1, \cdots, x_n) = \sum_{\nu=0}^{m} p_\nu(x_1, \cdots, x_n)$$

where each p_ν is homogeneous in x_1 and of degree ν in x_1; then $p_\nu \in Q$, $\nu = 0, 1, \cdots, m$.

The proof of this lemma is similar to that of [4, Lemma 3]. That is: let $\alpha_i \in \Omega$, then $p(\alpha_i x_1, \cdots, x_n) = \sum_\nu \alpha_i^\nu p_\nu(x_1, \cdots, x_n)$. Choosing m different elements $\alpha_1, \cdots, \alpha_m$ of Ω, one can find also $\lambda_{i\nu} \in Q$ such that $\sum_i \lambda_{i\nu} p(\alpha_i x_1, \cdots, x_n) = \Delta p_\nu(x_1, \cdots, x_n)$, where $\Delta = \prod_{i<k} (\alpha_i - \alpha_k)$. This implies $\Delta p_\nu(x) \in Q$; hence $p_\nu(x) \in Q$.

Lemma 3. *Let $p = p(x_1, \cdots, x_n)$ be a homogeneous polynomial in x_1. The polynomial p is quasi regular modulo Q if and only if p is nilpotent modulo Q.*

It is obvious that if p is nilpotent mod Q, then p is also quasi regular mod Q. Conversely, if p is quasi regular, then $p - q + pq \in Q$ for some polynomial q. Hence $q \equiv p + pq \equiv p + p^2 + pq \equiv \cdots \equiv p + p^2 + \cdots + p^{n+1} + p^{n+1}q$ (mod Q). Let $q = \sum_{\nu=0}^{m} q_\nu$ be the decomposition of q as a sum of homogeneous polynomials in x_1 of degrees $\nu = 0, 1, \cdots, m$. Thus $\bar{q} = \sum_1^{n+1} p^\mu + \sum_{\nu=0}^{m} p^{n+1} q_\nu - \sum_{\nu=0}^{m} q_\nu \in Q$. Now choose $n > m$. Then one observes that one of the elements of the decomposition of $\bar{q}$ as a sum of homogeneous polynomials in x_1 of different degrees must be p^n; hence by the preceding lemma it follows that $p^n \in Q$. q.e.d.

Theorem 2. *If S is a* PI-*ring without nilpotent ideals, then the universal ring $R' = \Omega[x]/Q_S$ is semisimple.*

Proof. Let $p(x_1, \cdots, x_n)$ be a polynomial belonging to the radical[2] of R'. Take as x_{n+1} any of the indeterminates of the set $\{x\}$ not appearing in $p(x)$. Evidently, $p(x)x_{n+1}$ also belongs to the radical. Since this polynomial is homogeneous in x_{n+1}, the preceding lemma yields $(p(x)x_{n+1})^m \in Q$ for some integer m. This evidently implies that if $s_1, \cdots, s_n$ are any elements of S, the ideal $p(s_1, \cdots, s_n)S$ is a nil ideal. From the results of [5] it follows that PI-rings which do not contain nilpotent ideals also do not contain right nil ideals; hence $p(s_1, \cdots, s_n)S = 0$. The absence of nilpotent ideals implies also that $p(s_1, \cdots, s_n) = 0$. Thus S satisfies the identity $p(x_1, \cdots, x_n) = 0$. By definition of Q_S, $p(x) \in Q_S$ and the proof is completed.

Applying the preceding result we are able to prove:

Theorem 3. *If Q is a nonzero T-ideal, then the radical of $\Omega|x|/Q$ is its maximal nil ideal.*

[2] This means, of course, "p is a representative of a class of the quotient $\Omega[x]/Q_S$ which belongs to the radical."

In fact, we show that the radical coincides with the lower radical [6]. Let J, L be the ideals of $\Omega[x]$ such that J/Q, L/Q are respectively the radical and the lower radical of $\Omega[x]/Q$. It is well known that $J\supseteq L\supseteq Q$. The quotient $\Omega[x]/L$ is a PI-ring[3] since it is a homomorphic image of $\Omega[x]/Q$, hence by Lemma 1, L contains the ideal of identities L_0 of this ring. Since $\Omega[x]/L\cong(\Omega[x]/Q)/(L/Q)$, $\Omega[x]/L$ does not contain nilpotent ideals; it follows, therefore, by the preceding theorem that $\Omega[x]/L_0$ is semisimple. Now $J/(J\cap L_0)$ $\cong(J, L_0)/L_0$. Since the first is an isomorphic image of the quasi regular ring J/Q, it is also quasi regular. But this implies that $(J, L_0)/L_0$ is quasi regular which is possible only if $(J, L_0)=L_0$, i.e., $L_0\supseteq J$. Thus $J\supseteq L\supseteq L_0\supseteq J$, which completes the proof.

The equality $J=L_0$ proves that:

Corollary 1. *The ideal J is a T-ideal.*

3. **The free ring** $\Omega[x]$. There is particular interest in the ideals of identities of the total matrix rings Ω_n of order n over Ω. Let M_n denote the ideal of identities of Ω_n, then we have $M_1\supseteq M_2\supseteq \cdots$. Each of the quotients $\Omega[x]/M_n$ is, by Theorem 2, a semi simple PI-ring. From [7, Theorem 2] we deduce that these rings are subdirect sums of central simple algebras of bounded degree. In the present case we can obtain the stronger result:

Lemma 4. *The quotient $\Omega[x]/M_n$ is a subdirect sum of total matrix rings of order n over Ω.*

Indeed, let τ be a homomorphism: $x_i\to r_i$ of $\Omega[x]$ onto Ω_n, where $\{r_i\}$ is a set of generators of Ω_n. Let Q_τ be the kernel of τ. Thus $\Omega[x]/Q_\tau\cong\Omega_n$. If τ ranges over all possible homomorphisms of this type, then $\bigcap Q_\tau=M_n$. For, by Lemma 1, it follows that[4] $Q_\tau\supseteq M_n$ for every τ. Hence $\bigcap Q_\tau\supseteq M_n$. Conversely, if $g(x_1, \cdots, x_n)\in\bigcap Q_\tau$, let $r_1, \cdots, r_n$ be any set of matrices of Ω_n. Let τ be the homomorphic mapping: $x_i\to r_i$, $i=1, \cdots, n$, and let the correspondence $x\to r$ for the rest of the set $\{x\}$ be defined so that the set $\{r\}$ will contain a set of generators of Ω_n. Then we obtain $g(r_1, \cdots, r_n)=0$. This proves that $g(x_1, \cdots, x_n)\in M_n$, and the proof is completed.

A subdirect sum of rings isomorphic with Ω_n is a subring of a complete direct sum of such rings. The latter is isomorphic with a total matrix ring of order n over a complete direct sum R of rings isomorphic with Ω. The ring R is known to be free of nilpotent elements since Ω is an integral domain. Hence:

[3] One readily verifies that L satisfies the assumption of the footnote 1.

[4] Q_τ evidently satisfies the assumption of footnote 1.

Corollary 2. *$\Omega[x]/M_n$ is isomorphic with a subring of a total matrix ring of order n over a commutative ring which does not contain nilpotent elements.*

It follows, therefore, by Theorem 1 that:

Theorem 4. *A ring S satisfies all identities of the total matrix ring Ω_n if and only if it is isomorphic with a subdirect sum of rings isomorphic with Ω_n; an alternative necessary and sufficient condition is that S be isomorphic with a subring of a total matrix ring of order n over a commutative ring.*

Lemma 5. $\bigcap M_n = 0$.

For if $g(x_1, \cdots, x_m) \in M_n$, then $g=0$ is satisfied by the ring of all finite matrices over Ω but no such identity exists by consequence 2 of [5]. Hence $g=0$, i.e., $\bigcap M_n = 0$.

It follows now in view of Lemma 4 that:

Corollary 3. *The free algebra $\Omega[x]$ is a subdirect sum of total matrix rings of finite order over Ω.*

We now apply the preceding result to the case $\Omega = I$, the ring of all integers. Since every ring is a homomorphic image of the free ring[5] $I[x]$, we obtain:

Theorem 4. *Every ring is a homomorphic image of a subdirect sum of total matrix rings of finite order over the ring of all integers.*

Since the ring I is a subdirect sum of prime fields, the preceding theorem yields:

Corollary 4. *Every ring is a homomorphic image of a subdirect sum of total matrix rings of finite order over prime fields.*

Consider the T-ideal N_n of $I[x]$ generated by the polynomial $x_1x_2 \cdots x_n$. This ideal contains all the polynomials with degree $\geqq n$. From consequence 2 of [5] we know that the minimal degree of the polynomials of the ideal M_k is greater than or equal to $2k$. Hence $N_n \supseteq M_k$ where $k = [(n+1)/2]$. Since every nilpotent ring is a homomorphic image of $I[x]/N_n$,[5] we obtain, by Theorem 3:

Theorem 5. *Every nilpotent ring of index of nilpotency n is a homomorphic image of a subdirect sum of total matrix rings over I of order bounded by $[(n+1)/2]$.*

By the arguments of Theorem 3 this readily implies:

[5] If the set $\{x\}$ is sufficiently large.

COROLLARY 5. *Every nilpotent ring of index of nilpotency n is homomorphic with a subring of a total matrix ring of order* $[(n+1)/2]$ *over a commutative ring which does not contain nilpotent elements.*

REMARK. If one considers algebras over a field F, then F can replace the integral domain I of all integers, and in this case the commutative rings obtained in Theorems 4, 5 and in Corollary 5 are either F or commutative algebras over F.

4. **The semisimple universal PI-rings.** The main object of the present section is to prove:

THEOREM 6. *The ideals* M_n *are the only T-ideals P of* $\Omega[x]$ *such that* $\Omega[x]/P$ *is semisimple.*[6]

To prove this theorem we need the following generalization of [**4**, Lemma 3]:

LEMMA 6. *Let C be a commutative ring with a unit element. The ideals of identities of a* PI-*ring S and of the direct product* $S\times C$ *(over any extension of* Ω*) coincide.*

The proof of this lemma was published in Hebrew in [**8**]. The following proof is simpler.

Put $x_j=\sum_i u_{ji}$, and write $g(\sum u_{1i},\cdots,\sum u_{ni})=\sum_k g_k(u_{ji})$ as a sum of homogeneous polynomials g_k. It follows now by Lemma 2 that the identities $g_k=0$ are satisfied by S. Let $x_j=a_j$, $a_j\in S\times C$, then $a_j=\sum_i s_{ji}c_{ji}$, $s_{ji}\in S$, $c_{ji}\in C$. Since the polynomials g_k are homogeneous, it follows that $g_k(s_{ji}c_{ji})=g_k(s_{ji})c=0$, where c is an element of C. By substituting $u_{ji}=s_{ji}c_{ji}$ we immediately obtain that $g(a_1,\cdots,a_n)=0$. q.e.d.

The ideal M_n is evidently also the ideal of identities of the total matrix ring R_n of order n over any commutative ring R which possesses a unit element. Now if A is any central simple algebra of order n^2 over its center F, one can find an extension $\overline{F}$ of F such that $A\times\overline{F}\cong\overline{F}_n$. In view of these facts, the preceding lemma implies that the ideal of identities of any central simple algebra A of order n^2 over its center is also M_n. With the aid of this result we can now prove Theorem 6.

Let P be a T-ideal such that $\Omega[x]/P$ is semisimple. Let k be the minimal degree of the polynomials of P. By [**7**, Theorem 2] it follows that this ring is a subdirect sum of central simple algebras A_m such

[6] This is equivalent to the fact that the only universal semisimple PI-rings are the quotients $\Omega[x]/M_n$.

that n^2 is the upper bound of their orders over their centers. Since the ideal of identities of these algebras are the ideals M_i, it follows readily that the ideal of identities of the subdirect sum is M_n. Thus, Lemma 1 yields that $P=M_n$, q.e.d.

Let S be a PI-ring. By Theorem 6 and by Corollary 1 it follows that the radical of the universal ring $\Omega[x]/Q_S$ is the quotient ideal M_n/Q_S.[7] From the main result of [2] it follows that M_n, $n\geqq 1$, contains the standard polynomial $S_{2n}(x)=\sum \pm x_{i_1}\cdots x_{i_{2n}}$ where the sum ranges over all permutations (i) of $2n$ letters and where the sign is taken positive for even permutations and negative for odd permutations. We shall use also the notation: $S_0(x)=x_1$. Applying Theorem 3 we now obtain that $S_{2n}(x)^m \in Q_S$ for some integer m. Hence:

COROLLARY 6. *Every PI-ring satisfies the identity* $S_{2n}(x)^m=0$.

The preceding corollary was proved under the assumption that Ω is infinite; nevertheless, it is true for every PI-ring. For if Ω is a finite integral domain, we consider the ring $S[t]$ of all polynomials over S in a commutative indeterminate t. Since S satisfies linear identities [4, Lemma 2], $S[t]$ also satisfies these identities. This ring can be considered as a PI-ring over the infinite integral domain $\Omega[t]$. Hence, by the preceding corollary, $S[t]$, and therefore also S, satisfy the identity $S_{2n}(x)^m=0$.

Denote by Σ_n the T-ideal generated by the standard polynomial $S_{2n}(x_1,\cdots,x_{2n})$. That is, Σ_n contains the polynomial of the form

$$h(x)=\sum a(x)S_{2n}(f_1(x),\cdots,f_{2n}(x))b(x). \tag{I}$$

From the main theorem of [2] we readily deduce that the total matrix ring Ω_n satisfies the identities $h(x)=0$, $h(x)\in\Sigma_n$. Naturally, this suggests the question whether the polynomials of Σ_n are the only polynomial identities satisfied by Ω_n. In the notation of the present paper this problem is equivalent to the assertion of the equality $M_n=\Sigma_n$.

From [2] it follows that $M_n\supseteq\Sigma_n$. On the other hand, consequence 3 of [5] implies that the minimal degree of the polynomials of M_{n+1} is greater than or equal to $2(n+1)$. Thus, the results of Corollary 1, Theorems 6 and 3 imply that M_n/Σ_n is the maximal nil ideal of $\Omega[x]/\Sigma_n$. This yields the following characterization of the polynomial identities satisfied by Ω_n (i.e., the polynomials which belong to M_n):

THEOREM 7. *The identity* $g(x)=0$ *is satisfied by* Ω_n *if and only if for*

[7] We use the notation $M_0=\Omega[x]$.

some integer m and for some indeterminate x_k not appearing in $g(x)$, the polynomial $(g(x)x_k)^m$ has the form (I).

This theorem provides only a partial answer to the problem of determining the identities satisfied by Ω_n. We conclude with the remark that the preceding theorem reduces the proof of a positive answer to the question raised above to the assertion that the quotient ring $\Omega[x]/\Sigma_n$ does not contain nilpotent elements. This is the case for $n=1$, since $\Omega[x]/\Sigma_1$ is a commutative ring without zero divisors, hence $M_1=\Sigma_1$. We conjecture that this holds for every n.

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HEBREW UNIVERSITY