A UNIQUENESS THEOREM FOR ORDINARY DIFFERENTIAL EQUATIONS INVOLVING SMOOTH FUNCTIONS

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In recent years Zygmund [2] has revived interest in a class of continuous real-valued functions of real variables described by a restriction on the second difference rather than on the first difference as in the more common Lipschitz conditions. The function class, called uniformly "smooth" (or class Λ^*) by Zygmund [2, p. 47], contains Lip 1 but is contained in Lip α for every $\alpha<1$. Smooth functions play an important role in the theory of trigonometric approximation where they, rather than Lip 1, appear to be the natural limiting class of Lip α as α approaches 1.

We here point out that, for ordinary differential equations involving smooth functions, there exists a unique solution through each initial point. This theorem generalizes the classical Lipschitz uniqueness theorem, see [1, p. 98], but is actually a direct consequence of Osgood's uniqueness theorem, see [1, p. 100].

THEOREM. *Let S: $dy/dx=f(x, y)$, where $f(x, y)$ is continuous in an open plane set R and suppose for each point $(x_0, y_0)\in R$ there exists a neighborhood $N(x_0, y_0)\subset R$ and a constant $k>0$ such that*

$$\left| f(x, y+h) + f(x, y-h) - 2f(x, y) \right| \leqq kh$$

for $(x, y)\in N(x_0, y_0)$ and all small $h>0$. Then there exists a unique solution $y(x; x_0, y_0)$ of S with $y(x_0; x_0, y_0)=y_0$ and $\partial y(x; x_0, y_0)/\partial x$ is defined and continuous in an open set of 3-space.

PROOF. By a slight modification of Zygmund's proof [2, p. 52] we obtain $|f(x, y+h)-f(x, y)| <k_1 h \log 1/h$ for all $(x, y)\in N(x_0, y_0)$, $k_1>0$, and all small $h>0$. But since $\int_{0+}dh/(h \log 1/h)=\infty$, Osgood's theorem yields the conclusions of the theorem.

BIBLIOGRAPHY

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2. A. Zygmund, *Smooth functions*, Duke Math. J. vol. 12 (1945) pp. 47–76.

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