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A CONVEX METRIC WITH UNIQUE SEGMENTS

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1. **Introduction.** If $D(x, y)$ is a convex metric for a continuous curve M (compact, locally connected, metric continuum), then for each pair of points p, q of M there is an arc pq in M from p to q such that pq is isometric to a straight line interval. We shall call it a segment.

Each continuous curve has a convex metric [1; 3; 5; 6; 2]. However, if $D(x, y)$ is a convex metric for M , there may be two segments from p to q . If $M = S_2$ is the surface of a sphere, $D(x, y)$ is the size of the central angle subtended by x and y , and p, q are diametrically opposite points, then there are many segments from p to q . In fact, we show in §4 that if $D(x, y)$ is any convex metric whatsoever for S_2 , each point of S_2 belongs to a pair of points which are not joined by a unique segment.

There is a dense subset W of S_2 such that no two points of W are diametrically opposite. If $D(x, y)$ is the previously mentioned convex metric for S_2 , then each pair of points of W are joined by one and only one segment. We shall show that for any continuous curve there is such a dense subset and such a convex metric.

THEOREM. *Each continuous curve M has a dense subset W and a convex metric $D(x, y)$ such that each pair of points of W belongs to a unique segment.*

We shall prove this result in much the same manner that it was shown that any continuous curve can be convexified. The metric is

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an adaption and modification of that described in Theorem 4 of [5]. We consider a decreasing sequence of core partitionings of M , assign sizes to the pieces of the partitionings, and define the distance between two points in terms of the sizes of the chains joining them. The convex metric $D(x, y)$ is described in §2 and the dense set W is given in §3.

We recall the following definitions. A *partitioning* of M is a finite collection of mutually exclusive connected open subsets of M whose sum is dense in M . A sequence of partitionings is decreasing if for each integer i , G_{i+1} is a refinement of G_i (each element of G_{i+1} lies in an element of G_i) and the mesh of G_i (maximum of diameters of elements of G_i) approaches 0 as i increases without limit.

A partitioning is *regular* if each of its pieces is the interior of its closure. If the partitioning H is a refinement of the regular partitioning K , the elements of H which have a boundary point in common with a boundary of an element of K are called *border elements*. Other elements are interior elements.

We call H a *core refinement* of K if each is regular, H is a refinement of K , each border element of H is adjacent to an interior element, and the sum of the interior elements of H in each element of K has a connected closure.

2. A convex metric for M . In this section we describe a convex metric $D(x, y)$ for M . We find in §3 that there is a dense subset W of M such that under $D(x, y)$, each pair of points of W belong to a unique segment.

Description of convex metric $D(x, y)$. It was shown in [3] that M has a decreasing sequence G_1, G_2, \dots of regular partitionings such that each is a core refinement of the preceding and each element of G_{i+1} is of diameter less than one-third the distance between any two nonadjacent elements of G_i .

Suppose the elements of the partitioning G_i are ordered $g_{i1}, g_{i2}, \dots, g_{in_i}$. We assign sizes to the elements of G_i as follows. The element g_{1j} of G_1 is assigned a size $1/4^j$. Each interior element g_{2j} of G_2 is assigned a size $1/4^{j+n_1}$ where n_1 is the number of elements in G_1 . If g_{2j} is a border element of G_2 , its size is one-half the size of the element of G_1 containing it plus $1/4^{j+n_1}$. In general, the size of g_{ij} is the first or second of the following expressions according as g_{ij} is an interior or a border element of G_i .

$$\text{Size } (g_{ij}) = \begin{cases} 1/4^{j+n_1+n_2+\dots+n_{i-1}} & \text{or} \\ 1/4^{j+n_1+n_2+\dots+n_{i-1}} + 1/2 \text{ size of element of} \\ & G_{i-1} \text{ containing } g_{ij}. \end{cases}$$

Had we not been interested in building a convex metric with unique segments, we would have made the following simplification in the definition of the sizes of the elements of G_i : each interior element of G_i would be given a size $1/(2^i \cdot n_i)$ and each border element of G_i would be given a size of one-half the size of the element of G_{i-1} containing it.

If g is an element of G_i , we denote its size by $S_i(g)$. If p is a point of g , we shall also denote $S_i(g)$ by $S_i(p)$. If G is a collection of elements of G_i , the sum of the sizes of the elements of G is denoted by $S_i(G)$.

Suppose K is a continuum which is the closure of the sum of a sub-collection of G_i . The sum of the sizes of the elements of G_i in K is called the i th size of K and is denoted by $S_i(K)$.

If p and q are two points of M , we define $E_i(p, q)$ to be the minimum of all numbers $S_i(K)$ where K is a continuum with i th size which contains $p+q$. We may regard $E_i(p, q)$ as the i th approximation to the distance between p and q . Then $D(p, q) = \lim E_i(p, q)$.

Existence of $\lim E_i(p, q)$. Suppose K is a continuum containing $p+q$ and having an i th size of $E_i(p, q)$. It is the closure of the sum of a chain of elements of G_i such that p and q lie in the closures of the end links of this chain. There is a continuum K' in K containing $p+q$ such that K' has an $(i+1)$ st size but does not contain three border elements of G_{i+1} in the same element of G_i . Then $S_{i+1}(K') \leq S_i(K) + 1/4^{1+n_1+\dots+n_i} + \dots + 1/4^{n_1+n_2+\dots+n_{i+1}}$. Therefore

$$(1) \quad E_{i+1}(p, q) + 1/(3 \cdot 4^{n_1+n_2+\dots+n_{i+1}}) \leq E_i(p, q) + 1/(3 \cdot 4^{n_1+n_2+\dots+n_i}).$$

Now inequality (1) implies that $\{E_i(p, q) + 1/(3 \cdot 4^{n_1+n_2+\dots+n_i})\}$ is a monotone nonincreasing sequence of positive numbers. Hence $\lim E_i(p, q)$ exists.

In fact, $E_i(p, q)$ converges to $D(x, y)$ uniformly since it can be shown that $|D(p, q) - E_i(p, q)| < 2/2^i$.

That $D(x, y)$ is a convex metric for M follows by an argument similar to that used in [3] to show that a continuous curve has a convex metric.

3. Unique segments in M . In this section we show that if M is given the convex metric $D(x, y)$ described in the last section, there is a dense subset W of M such that each pair of points of W belongs to a unique segment.

Description of set W . Let G_1, G_2, \dots be the decreasing sequence of core partitionings of M described in the last section. Let C_i be the sum of the core elements of G_i and W_i be the intersection of C_i, C_{i+1} ,

C_{i+2}, \dots . Then W_i is a closed set that intersects each element of G_{i-1} . We shall show that each pair of points of $W = W_2 + W_3 + \dots$ belongs to a unique segment in M .

A unique shortest i th chain from p to q . If g and g' are two elements of G_i , the denominators of $S_i(g)$ and $S_i(g')$ will lie between $4^{1+n_1+\dots+n_{i-1}}$ and $4^{n_1+n_2+\dots+n_i}$ inclusive but they will not be equal. Therefore, if F_1 and F_2 are two subcollections of G_i ,

$$(2) \quad |S_i(F_1) - S_i(F_2)| \geq 1/4^{n_1+n_2+\dots+n_i}.$$

Hence, for each pair of points p, q there is a unique continuum K containing $p+q$ such that $S_i(K) = E_i(p, q)$.

Finding a continuum H of i th size near a continuum K of $(i+1)$ st size. If K is a continuum with an $(i+1)$ st size and g_1, g_2 are two elements of G_i intersecting K , there is a continuum H with an i th size such that H contains g_1+g_2 , H is adjacent to each element of G_i that is adjacent to K , and

$$(3) \quad S_i(H) < S_{i+1}(K) + S_i(g_1)/2 + S_i(g_2)/2.$$

We obtain the continuum H in the following fashion. Let G' be the collection of border elements of G_{i+1} that lie in K and G'' be the collection of all elements g of G' such that g lies in neither g_1, g_2 , nor any element of G_i containing two elements of G' . Then G'' is the sum of two mutually exclusive subcollections G_1'', G_2'' such that each element of G'' is adjacent to an element of $(G' - G'') + G_1''$ and also to an element of $(G' - G'') + G_2''$. Let G_k'' be the one of G_1'', G_2'' such that the sum of the $(i+1)$ st sizes of its elements is the smaller. Then H is the closure of the sum of all elements g of G_i such that g contains an element of $(G' - G'') + G_k''$.

Since each element of G' is adjacent to H and the mesh of G_{i+1} is less than the distance between nonadjacent elements of G_i , each element of G_i that is adjacent to K is also adjacent to H .

Let F' be the collection of elements of G_i which contain an element of $G' - G''$ and F'' be the collection of those that contain an element of G_k'' . Since $S_i(F'') < S_{i+1}(G'')$ and $S_i(F') < S_{i+1}(G' - G'') + S_i(g_1) + S_i(g_2)$, inequality (3) follows.

Unless the collection of elements of G_i that irreducibly cover the sum of the elements of G_1'' is the same as those that irreducibly cover the sum of the elements of G_2'' , it follows from inequality (2) that $S_i(F'') + 1/(2 \cdot 4^{n_1+n_2+\dots+n_i}) < S_{i+1}(G'')$. Hence, unless H contains K , we obtain the following strong form of inequality (3):

$$(4) \quad S_i(H) + 1/(2 \cdot 4^{n_1+n_2+\dots+n_i}) < S_{i+1}(K) + S_i(g_1)/2 + S_i(g_2)/2.$$

Shortest (i+1)st chain lies in shortest ith chain. Suppose p and q are two points of W_i and H, K are the two continua such that each contains $p+q$, $S_i(H) = E_i(p, q)$, and $S_{i+1}(K) = E_{i+1}(p, q)$. We shall show that K lies in H .

There is a continuum K' in H containing $p+q$ such that $S_{i+1}(K') \leq S_i(H) - S_i(p)/2 - S_i(q)/2 + 1/4^{1+n_1+\dots+n_i} + \dots + 1/4^{n_1+n_2+\dots+n_{i+1}}$. Therefore

$$(5) \quad S_{i+1}(K) \leq S_i(H) - S_i(p)/2 - S_i(q)/2 + 1/(3 \cdot 4^{n_1+n_2+\dots+n_i}) - 1/(3 \cdot 4^{n_1+n_2+\dots+n_{i+1}}).$$

Now it follows from inequality (3) that there is a continuum H' containing $p+q$ such that $S_i(H') < S_{i+1}(K) + S_i(p)/2 + S_i(q)/2$. This and inequality (5) implies that $S_i(H') - S_i(H) < 1/4^{n_1+n_2+\dots+n_i}$ and inequality (2) shows that $H' = H$.

Unless K lies in $H' = H$, it follows from inequality (4) that $S_i(H) + 1/(2 \cdot 4^{n_1+n_2+\dots+n_i}) < S_{i+1}(K) + S_i(p)/2 + S_i(q)/2$. This is impossible because of inequality (5). Therefore K is a subset of H .

The unique segment. Suppose p and q are two points of W and K_i is the continuum containing $p+q$ such that $S_i(K_i) = E_i(p, q)$. If m is an integer such that p and q belong to core elements of G_n if $n \geq m$, we find that $K_m \supset K_{m+1} \supset K_{m+2} \supset \dots$. In fact, since the chain of closures of elements of G_{m+i+1} whose sum is K_{m+i+1} runs straight through the chain of closures of elements of G_{m+i} whose sum is K_{m+i} without doubling back and zigzagging, $K_m \cdot K_{m+1} \cdot \dots$ is an arc pq from p to q . We show that pq is a unique segment from p to q by showing that any point not on pq is not between p and q .

Suppose r is a point between p and q but not on pq . Let H_i be the sum of the closures of all elements of G_i adjacent to K_i . Then there is an integer t greater than m such that r does not belong to H_t . Furthermore, for each positive number ϵ there is a positive integer s and a continuum R_s containing $p+r+q$ such that $S_{t+s}(R_s) < D(p, q) + \epsilon$. We show that there is no such point r by showing that there is no integer s for $\epsilon = 2/(3 \cdot 4^{n_1+n_2+\dots+n_t})$.

The continuum R_s is adjacent to $M - H_t$. By inequality (3) there is a continuum R_{s-1} containing $p+q$ such that R_{s-1} is adjacent to $M - H_t$ and

$$S_{t+s-1}(R_{s-1}) < S_{t+s}(R_s) + S_{t+s-1}(p)/2 + S_{t+s-1}(q)/2 < D(p, q) + \epsilon + S_{t+s-1}(p)/2 + S_{t+s-1}(q)/2.$$

Furthermore, there is a continuum R_{s-2} containing $p+q$ such that R_{s-2} is adjacent to $M - H_t$ and

$$\begin{aligned}
 S_{t+s-2}(R_{s-2}) &< S_{t+s-1}(R_{s-1}) + S_{t+s-2}(p)/2 + S_{t+s-2}(q)/2 \\
 &< D(p, q) + \epsilon + S_{t+s-1}(p)/2 + S_{t+s-1}(q)/2 \\
 &\quad + S_{t+s-2}(p)/2 + S_{t+s-2}(q)/2.
 \end{aligned}$$

Continuing in this manner we find that there is a continuum R_0 containing $p+q$ such that R_0 is adjacent to $M-H_t$ and

$$\begin{aligned}
 (6) \quad S_t(R_0) &< D(p, q) + \epsilon \\
 &\quad + S_t(p)/2 + S_{t+1}(p)/2 + \dots + S_{t+s-1}(p)/2 \\
 &\quad + S_t(q)/2 + S_{t+1}(q)/2 + \dots + S_{t+s-1}(q)/2.
 \end{aligned}$$

Since K_t is not adjacent to $M-H_t$, $R_0 \neq K_t$ and it follows from inequality (2) that

$$(7) \quad S_t(K_t) + 1/4^{n_1+n_2+\dots+n_t} \leq S_t(R_0).$$

It follows from inequality (5) that

$$\begin{aligned}
 (8) \quad D(p, q) &\leq E_t(p, q) - S_t(p)/2 - S_{t+1}(p)/2 - \dots \\
 &\quad - S_t(q)/2 - S_{t+1}(q)/2 - \dots + 1/(3 \cdot 4^{n_1+n_2+\dots+n_t}).
 \end{aligned}$$

But inequalities (7), (6), and (8) imply that

$$\begin{aligned}
 1/4^{n_1+n_2+\dots+n_t} &< \epsilon - S_{t+s}(p)/2 - S_{t+s+1}(p)/2 - \dots \\
 &\quad - S_{t+s}(q)/2 - S_{t+s+1}(q)/2 - \dots \\
 &\quad + 1/(3 \cdot 4^{n_1+n_2+\dots+n_t}).
 \end{aligned}$$

Hence, if we take $\epsilon = 2/(3 \cdot 4^{n_1+n_2+\dots+n_t})$, we find that the supposition that there is a point r between p and q which is not on pq leads to a contradiction.

4. Examples and questions. Even though a continuous curve has a convex metric, this metric may be quite different from a Euclidean metric. For example, if S is a horizontally based square plus its interior in the plane and $D(p, q)$ is defined to be the sum of the absolute values of the difference of the ordinates and the difference of the abscissas of p and q , each point of S lies on some segment between opposite vertices of S . It is of interest to get convex metrics that resemble Euclidean metrics.

Continuous curves each of whose segments is unique. If I^n is a cube plus its interior in Euclidean n -space, its Euclidean metric causes each pair of its points to belong to a unique segment. Also, if M is a continuous curve each of whose nondegenerate cyclic elements is topologically a cube of some dimension, M has such a convex metric. Hence, each unicoherent plane continuous curve has such a metric.

If C is a cone whose base is a figure eight, C may be regarded as the sum of two closed discs that are sewn together along a radius of each. This shows that I^2 is not the only continuous curve which is 2-dimensional at each of its points and has a convex metric under which all segments are unique.

Suppose M is a continuous curve with a convex metric under which each pair of points belongs to a unique segment. Then for each point p of M there is a continuous transformation $F(m, t)$ (m element of M , t element of $[0, 1]$) of the cartesian product of M and a straight line interval onto M such that $F(m, 1) = m$, $F(m, 0) = p$, $F(F(m, t_1), t_2) = F(m, t_1 + t_2)$. By letting $t_1 = 0$ we find that the last equation implies that $F(p, t) = p$. We can define $F(m, t)$ as the point which divides the segment from p to m in the same ratio that t divides the interval from 0 to 1.

The interior of each sphere of M is contractible to its center.

Questions. Does the existence of such a transformation $F(m, t)$ guarantee that M can be assigned a convex metric under which segments are unique?

Not only would it be interesting to know a topological characterization of continuous curves which have convex metrics with only unique segments but it would also be interesting to know which continuous curves have the following type of convex metric. If two segments in Euclidean space intersect in more than one point, their sum is a segment. If an n -dimensional continuous curve has such a convex metric, is it necessarily topologically equivalent to an n -cube I^n ?

The surface of a sphere has a convex metric under which nearby pairs of points belong to a unique segment. One might wonder if the local structure of a continuous curve determines whether or not it has such a convex metric—that is, does a continuous curve M have such a metric if for each point p of M there is an open subset N_p of M containing p such that N_p is topologically equivalent to a set with such a metric?

Continuous curves with nonunique segments. The surface of a sphere S^n in Euclidean $(n+1)$ -space has the property that if p is one of its points there is a point q of it such that there is not a unique segment from p to q . We know this because S^n is not contractible—that is, it cannot be shrunk to a point [4]. Similarly, a continuous curve which is not unicoherent is not contractible [7] and therefore does not have a unique segment between each pair of its points.

Not each contractible continuous curve has a convex metric with only unique segments. Suppose that in the plane J_i ($i=1, 2, \dots$) is a circle with radius $1/i$ and center at $(1/i, 0)$. If C is a cone with

$J_1 + J_2 + \dots$ as a base, it is contractible but it does not have a convex metric with only unique segments because there is no contractible neighborhood not containing the vertex about the point $(0, 0)$ of the base.

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