A NOTE ON PAIRS OF NORMAL MATRICES
WITH PROPERTY L

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The following is a generalization of a theorem due to Motzkin and Taussky [1] concerning matrices with property L. By definition, two matrices $A$ and $B$ have property L if any linear combination $\alpha A + \beta B$ (where $\alpha$ and $\beta$ are complex numbers) has as characteristic roots the numbers $\alpha \lambda_i + \beta \mu_i$ where the $\lambda_i$ are the characteristic roots of $A$ and the $\mu_i$ are the characteristic roots of $B$ both taken in a special ordering. It is shown in the above paper that if two hermitian matrices have property L, they commute.

The following lemma may be noted:

**Lemma.** If a normal matrix has its characteristic roots in the main diagonal, then the matrix is diagonal.

Let $A$ be normal with characteristic roots $\alpha_1, \alpha_2, \ldots, \alpha_n$. Then there exists a unitary matrix $U$ such that $UA^U = D$ where $D$ is diagonal with the characteristic roots of $A$ appearing in the diagonal. Then $UA^U = D^U$ and $UA^U A^U = D^U$ and the characteristic roots of $A A^U$ are $|\alpha_i|^2$. Since the sum of the diagonal elements of $AA^U$ is equal to the sum of its characteristic roots, all nondiagonal elements of $A$ must be zero if the characteristic roots of $A$ appear along the diagonal of $A$.

At this point the following theorem (Theorem 1) from [1] is recalled: Let the $n$-rowed square matrices $A$ and $B$ have property L. Let $t$ be the number of different characteristic roots of $A$ and assume that all the characteristic roots $\lambda_i$ of $A$ are arranged in sets of equal roots. Let $m_i$ be the multiplicity of the characteristic root $\lambda_i$ of $A$ and assume there are $m_i$ independent characteristic vectors corresponding to each $\lambda_i$. Let $B_i = P^{-1}BP$ where $A_i = P^{-1}AP$ is in Jordan normal form. Then $B_i = (B_{ij})$, $i, j = 1, \ldots, t$, where $B_{ij}$ is an $m_i$-rowed square matrix $(i=1, \ldots, t)$ and $|\lambda I - B_i| = \prod_{i=1}^t |\lambda I - B_{ii}|$.

**Theorem.** If $A$ and $B$ are normal matrices with property L, they commute.

Let the matrix $A$ be brought into diagonal form by an appropriate unitary matrix $U$ such that $UA^U = D$ is diagonal where like roots are grouped together. Let

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UBV^{CT} = B_1 = (B_{ij}) (i, j = 1, \cdots, k) where each $B_{ij}$ has an order equal to that of the number of like roots in the corresponding diagonal position in $D$. By the above-mentioned theorem the roots of $B_1$ are, in totality, the roots of $B_{11}, B_{22}, \cdots, B_{kk}$ taken together. From Schur [2] it is known that any square matrix with complex elements can be brought into a triangular matrix under a unitary transformation. Let $U_i$ be a unitary matrix such that $U_i B_{ii} U_i^{CT} = T_i$ where each $T_i$ is a triangular matrix. Let

$$V = \begin{bmatrix} U_1 & 0 & \cdots & 0 \\ 0 & U_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & U_k \end{bmatrix}.$$ 

Then $VDV^{CT} = D$ and $VB_i V^{CT}$ is a normal matrix with the triangular matrices $T_i$ in the main diagonal and consequently its characteristic roots down the main diagonal. Hence $VB_i V^{CT}$ is diagonal because of the lemma and it follows that $A$ and $B$ commute.

**Bibliography**


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