REFERENCES

1. Eckford Cohen, *Sums of an even number of squares in* $GF[p^n, x]$. II, Duke Math. J. vol. 14 (1947) pp. 543–557.

2. André Weil, *Numbers of solutions of equations in finite fields*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 497–508.

DUKE UNIVERSITY

CONGRUENCES CONNECTED WITH THREE-LINE LATIN RECTANGLES

L. CARLITZ

1. Introduction. In a recent paper [1], J. Riordan set up the recurrences

$$K_n = n^2K_{n-1} + (n)_2K_{n-2} + 2(n)_3K_{n-3} + k_n, \tag{1.1}$$

where $(n)_r = n(n-1) \cdots (n-r+1)$, and

$$k_n + nk_{n-1} = -(n-1)2^n; \tag{1.2}$$

here $K_n = K(3, n)$, the number of reduced three-line latin rectangles. He also proved the congruences

$$k_{n+p} \equiv 2k_n, \qquad K_{n+p} \equiv 2K_n \pmod{p}, \tag{1.3}$$

where p is a prime >2.

In the present note we shall extend (1.3). We show first that for arbitrary m,

$$k_{n+m} \equiv 2^mk_n, \qquad K_{n+m} \equiv 2^mK_n \pmod{m}. \tag{1.4}$$

More generally if we define

$$\Delta f(n) = f(n+m) - 2^mf(n), \qquad \Delta^s f(n) = \Delta\Delta^{s-1}f(n) \tag{1.5}$$

for fixed $m \geqq 1$, then

$$\Delta^r k_n \equiv 0 \equiv \Delta^r K_n \pmod{m^r} \tag{1.6}$$

for all $r \geqq 1$.

2. Proof of (1.4). In (1.2) replace n by $n+m$ so that

Presented to the Society, June 21, 1952; received by the editors April 16, 1952.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

$$k_{n+m} + (n+m)k_{n+m-1} = -(n+m-1)2^{n+m}.$$

Comparison with (1.2) yields

$$(k_{n+m} - 2^m k_n) + n(k_{n+m-1} - 2^m k_{n-1}) + m(k_{n+m-1} + 2^{n+m}) = 0$$

or more briefly, using (1.5),

$$\Delta k_n + n\Delta k_{n-1} + m(k_{n+m-1} + 2^{n+m}) = 0. \tag{2.1}$$

Clearly (2.1) implies

$$\Delta k_n \equiv (-1)^n n! \Delta k_0 \pmod{m}.$$

Since by (1.2)

$$\Delta k_0 = k_m - 2^m k_0 \equiv 2^m - 2^m k_0 \equiv 0 \pmod{m},$$

it follows at once that $\Delta k_n \equiv 0 \pmod{m}$. This proves the first half of (1.4).

In the next place, using (1.1) we get

$$K_{n+m} \equiv n^2 K_{n+m-1} + (n)_2 K_{n+m-2} + 2(n)_3 K_{n+m-3} + k_{n+m}, \tag{2.2}$$

so that

$$\Delta K_n \equiv n^2 \Delta K_{n-1} + (n)_2 \Delta K_{n-2} + 2(n)_3 \Delta K_{n-3}. \tag{2.3}$$

Since by (1.1)

$$K_m \equiv k_m, \qquad K_{m+1} \equiv K_m + k_{m+1} \equiv k_m + k_{m+1} \equiv 0,$$
$$K_{m+2} \equiv 4K_{m+1} + 2K_m + k_{m+2} \equiv 2k_m + k_{m+2} \equiv 0$$

(using the special values $k_0 = 1$, $k_1 = -1$, $k_2 = -2$, $K_0 = 1$, $K_1 = K_2 = 0$), it follows that

$$\Delta K_0 \equiv \Delta K_1 \equiv \Delta K_2 \equiv 0 \pmod{m}. \tag{2.4}$$

Clearly (2.3) and (2.4) imply

$$\Delta K_n \equiv 0 \pmod{m} \tag{2.5}$$

for all $n \geqq 0$. This completes the proof of (1.4).

3. **Proof of (1.6).** We shall require an extension of (2.1). Replacing n by $n+m$, we get

$$\Delta^2 k_n + n\Delta^2 k_{n-1} + 2m\Delta k_{n+m-1} = 0,$$

and it is then easy to get the general formula

$$\Delta^r k_n + n\Delta^r k_{n-1} + rm\Delta^{r-1} k_{n+m-1} = 0 \qquad (n \geqq 0) \tag{3.1}$$

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

for $r \geqq 2$. We now use (3.1) to prove

$$\Delta^r k_n \equiv 0 \pmod{m^r} \qquad (r \geqq 1). \tag{3.2}$$

Indeed we have already proved (3.2) for the value $r=1$. If then we assume (3.2) for the value $r-1$, (3.1) implies

$$\Delta^r k_n \equiv -n\Delta^r k_{n-1} \equiv (-1)^n n!\Delta^r k_0. \tag{3.3}$$

Now if we take $n=0$ in (3.1) we get

$$\Delta^r k_0 = -rm\Delta^{r-1}k_{m-1} \equiv 0,$$

by the inductive hypothesis. Hence (3.3) reduces to (3.2). This proves the first half of (1.6).

We now prove

$$\Delta^r K_n \equiv 0 \pmod{m^r} \qquad (r \geqq 1). \tag{3.4}$$

By (2.5), (3.4) holds for $r=1$; we therefore assume that it holds for the value $r-1$. Now it follows from (1.1) that

$$\begin{aligned}\Delta^r K_n = {} & n^2\Delta^r K_{n-1} + (n)_2\Delta^r K_{n-2} + 2(n)_3\Delta^r K_{n-3} + \Delta^r k_n \\ & + r\{(\Delta n^2)\Delta^{r-1}K_{n+m-1} + (\Delta(n)_2)\Delta^{r-1}K_{n+m-2} \\ & + 2(\Delta(n)_3)\Delta^{r-1}K_{n+m-3}\} + \cdots\end{aligned} \tag{3.5}$$

for all $n \geqq 0$. Since $\Delta^s n^k \equiv 0 \pmod{m^s}$ for $k \geqq 0$, it is evident from (3.5) that we need merely examine $\Delta^r K_n$ for $n=0, 1, 2$. In the first place (3.5) implies $\Delta^r K_0 \equiv 0 \pmod{m^r}$ by the inductive hypothesis. Secondly for $n=1$, we see that $\Delta^r K_1 \equiv \Delta^r K_0 \equiv 0$, and for $n=2$, $\Delta^r K_2 \equiv 4\Delta^r K_1 + 2\Delta^r K_0 \equiv 0$. Thus $\Delta^r K_n \equiv 0$ for all $n \geqq 0$. This completes the proof of (1.6).

Reference

1. John Riordan, *A recurrence relation for three-line latin rectangles*, Amer. Math. Monthly vol. 59 (1952) pp. 159–162.

Duke University

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use