ON CONTINUITY PROPERTIES OF DERIVATIVES OF SEQUENCES OF FUNCTIONS

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It is the purpose of this note to indicate some readily proved results concerning the pth derivatives of convergent sequences of functions of a real variable; these results are associated with repeated term-by-term differentiation, and involve especially values assumed, total variation, and modulus of continuity of pth derivatives.

As an illustration of this material, we remark that it is intuitively obvious that if a sequence of functions $f_n(x)$ possessing derivatives approaches the function $\cos x$ in the interval $-1 \leqq x \leqq +1$, not necessarily uniformly, then for n sufficiently large the function $f_n'(x)$ has at least one zero for some x near $x=0$; the same conclusion holds if the sequence $f_n(x)$ approaches the function $|x|$ in the interval $-1 \leqq x \leqq +1$.

Although the two investigations were undertaken independently, the present note has close connections with a forthcoming paper by Ulam and Hyers. The latter authors emphasize consequences of *uniform* convergence of a sequence, but under appropriate circumstances study the values taken on by, and especially the vanishing of, the pth derivatives of the functions of an approximating sequence; they also investigate analogous problems for functions of several variables.

THEOREM 1. *Let the functions $f_n(x)$ converge to the function $f(x)$ in the interval I: $a \leqq x \leqq b$, and let both $f_n(x)$ and $f(x)$ possess derivatives of order p (>0) at every point of I. Let there be given a point x_0 of I, and positive numbers δ and ϵ. Then there exists N such that for every $n > N$ the function $f_n^{(p)}(x)$ takes on a value $f_n^{(p)}(x_n)$ which satisfies*

$$(1) \qquad \left| f_n^{(p)}(x_n) - f^{(p)}(x_0) \right| < \epsilon$$

at some point x_n of the interval $|x - x_0| < \delta$.

Consider first the case $p=1$. At a suitably chosen point x_0' of I near x_0 we have

$$(2) \qquad \left| \frac{f(x_0') - f(x_0)}{x_0' - x_0} - f'(x_0) \right| < \frac{\epsilon}{2}, \qquad | x_0 - x_0' | < \delta.$$

For n sufficiently large we have by the convergence of the sequence

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$f_n(x)$ in the points x_0' and x_0

$$(3)\qquad \left|\frac{f_n(x_0') - f_n(x_0)}{x_0' - x_0} - \frac{f(x_0') - f(x_0)}{x_0' - x_0}\right| < \frac{\epsilon}{2}.$$

But the first fraction in (3) has the value $f_n'(x_n)$, where x_n is a suitably chosen point in the interval $|x-x_0| < \delta$, so inequality (1) for $p=1$ follows from (2) and (3).

It is now clear how the proof of Theorem 1 can be completed by induction. Assume the theorem true for the index $p-1$; we prove the theorem for the index p. We chose x_0' in I satisfying

$$(4)\qquad \left|\frac{f^{(p-1)}(x_0') - f^{(p-1)}(x_0)}{x_0' - x_0} - f^{(p)}(x_0)\right| < \epsilon, \qquad |x_0 - x_0'| < \delta.$$

The function $f^{(p-1)}$ possesses a derivative and hence is continuous in I, so the corresponding inequality is valid if in the denominator of the fraction the values x_0' and x_0 are replaced by arbitrary values X_0' and X_0 in suitable neighborhoods $N(x_0')$ and $N(x_0)$ of x_0' and x_0 respectively (these neighborhoods are to be chosen to lie in $|x-x_0| < \delta$), and if in the numerator of the fraction the values $f^{(p-1)}(x_0')$ and $f^{(p-1)}(x_0)$ are replaced by arbitrary values g' and g satisfying suitable inequalities

$$(5)\qquad |f^{(p-1)}(x_0') - g'| < \epsilon_1, \qquad |f^{(p-1)}(x_0) - g| < \epsilon_1.$$

That is to say, if X_0' and X_0 lie in $N(x_0')$ and $N(x_0)$, and if (5) is valid, then we have

$$(6)\qquad \left|\frac{g' - g}{X_0' - X_0} - f^{(p)}(x_0)\right| < \epsilon.$$

By Theorem 1 as assumed true for the index $p-1$, there exists N so that for $n > N$ the function $f_n^{(p-1)}(x)$ takes on a value g' satisfying (5) in some point X_0' of $N(x_0')$ and simultaneously takes on a value g satisfying (5) in some point X_0 of $N(x_0)$; here X_0' and X_0 naturally depend on n. For such values of n we have

$$\left|\frac{f_n^{(p-1)}(X_0') - f_n^{(p-1)}(X_0)}{X_0' - X_0} - f^{(p)}(x_0)\right| < \epsilon.$$

The fraction is equal to $f_n^{(p)}(x_n)$ in some point x_n between X_0' and X_0, so x_n lies in the interval $|x-x_0| < \delta$, and Theorem 1 is established.

We remark that at the end points of I we deal wholly with one-sided derivatives of $f(x)$ and $f_n(x)$; it follows that the prescribed interval for x_n may be restricted to a one-sided neighborhood also if

x_0 is an interior point of I.

Ulam and Hyers consider Theorem 1 in the case $f^{(p)}(x_0)=0$, where $f^{(p)}(x)$ changes sign at $x=x_0$, and require uniform convergence of the sequence $f_n(x)$; their method involves the use of mth differences, and can be combined with the present methods to establish Theorem 1.

Proof of the following is essentially contained in the discussion as given:

COROLLARY 1. *Under the hypothesis of Theorem* 1, *let* x_1 *and* x_2 $(<x_1)$ *lie in* I; *then for* n *sufficiently large there exist* X_1 *and* X_2 *in* I *depending on* n *such that*

$$\left|\frac{f_n^{(p)}(X_1)-f_n^{(p)}(X_2)}{X_1-X_2}-\frac{f^{(p)}(x_1)-f^{(p)}(x_2)}{x_1-x_2}\right|<\epsilon$$

for some points X_1 *and* X_2 *with* $|X_1-x_1|<\delta$, $|X_2-x_2|<\delta$. *In particular if* $f_n^{(p+1)}(x)$ *exists at every point of* I, *for* n *sufficiently large there exists some point* X, $x_2<X<x_1$, *such that we have*

$$\left|f_n^{(p+1)}(X)-\frac{f^{(p)}(x_1)-f^{(p)}(x_2)}{x_1-x_2}\right|<\epsilon.$$

If x_1 *and* x_2 $(<x_1)$ *are arbitrary points of* I, *and if we have* $f^{(p)}(x_1)>f^{(p)}(x_2)$ [*or* $<f^{(p)}(x_2)$], *and if* $f^{(p)}(x_1)>A>f^{(p)}(x_2)$ [*or* $f^{(p)}(x_1)<A<f^{(p)}(x_2)$], *then for* n *sufficiently large* $f_n^{(p)}(x)$ *takes on the value* A *in some point* X_n, $x_1<X_n<x_2$.

The last remark follows from Theorem 1 and the classical property of the derivative $f_n^{(p)}(x)$.

Both Theorem 1 and Corollary 1 are of significance in the study of approach by functions $f_n(x)$ having more derivatives than the limit function $f(x)$. The interval I of Theorem 1 may be only a subinterval of a larger interval of convergence. For instance suppose $f_n(x)\to f(x)\equiv|x|$ in I': $-1\leqq x\leqq 1$. Suppose $f_n'(x)$ exists at every point of I', and let $\delta(>0)$ be given. For n sufficiently large, it follows from Theorem 1 that $f_n'(x)$ takes a value near unity in a neighborhood of the point $\delta/2$ and takes a value near minus unity in a neighborhood of the point $-\delta/2$, hence that $f_n'(x)$ takes the value zero in some point of the interval $|x|\leqq\delta$; compare the second part of Corollary 1. If $f_n''(x)$ exists and is continuous in I', the equation

$$f_n'(x_1)-f_n'(x_2)=\int_{x_2}^{x_1}f_n''(x)dx,$$

where x_1 and x_2 are near $\delta/2$ and $-\delta/2$ respectively, shows that for

n sufficiently large, $f_n''(x)$ takes some value numerically greater than $1/\delta$ at some point of the interval $|x|<\delta$. Extension of this reasoning shows that if $f_n^{(p)}(x)$ exists and is continuous at every point of I', and if M and δ are arbitrary, then for n sufficiently large $f_n^{(p)}(x)$ takes some value numerically greater than M in the interval $|x|<\delta$. Of course Corollary 1 extends to higher difference quotients.

These remarks concerning approximation to the function $|x|$ are closely related to the more obvious fact that if $f(x)$ is defined throughout the interval I but is discontinuous at the point x_0 of I, and if the sequence of functions $f_n(x)$ continuous in I converges in I to $f(x)$, then if M and $\delta(>0)$ are given, for n sufficiently large the difference quotient of $f_n(x)$ is numerically greater than M at some point of the interval $|x-x_0|<\delta$; if $f_n'(x)$ exists throughout I, then for n sufficiently large $f_n'(x)$ is numerically greater than M at some point of the interval $|x-x_0|<\delta$; a similar conclusion applies to the higher derivatives of $f_n'(x)$ if they exist, for we cannot have here $f_n'(x)\to+\infty$ or $f_n'(x)\to-\infty$ in an interval, as is shown in Lemma 1 below. It follows similarly that if $f(x)$ is continuous in I but has no derivative at the point x_0 of I, if the sequence of functions $f_n(x)$ continuous in I converges in I to $f(x)$ and if $f_n'(x)$ exists in I, then if M and δ (>0) are given, for n sufficiently large the difference quotient of $f_n'(x)$ is numerically greater than M at some point of the interval $|x-x_0|<\delta$; if $f_n''(x)$ exists throughout I, then for n sufficiently large the second derivative $f_n''(x)$ is numerically greater than M at some point of the interval $|x-x_0|<\delta$; a similar conclusion applies to higher derivatives if they exist.

COROLLARY 2. *Under the conditions of Theorem* 1 *we have*

$$\liminf_{n\to\infty}\ [\text{Total variation of } f_n^{(p)}(x) \text{ in } I]$$
$$\geqq [\text{Total variation of } f^{(p)}(x) \text{ in } I].$$

Corollary 2 is a direct consequence of Theorem 1 and the definition of the total variation of $f^{(p)}(x)$ in I as

$$\text{l.u.b.} \sum_{k=0}^{K} \left| f^{(p)}(\xi_{k+1}) - f^{(p)}(\xi_k) \right|, \quad a=\xi_0<\xi_1<\cdots<\xi_K<\xi_{K+1}=b;$$

the proof is left to the reader.

COROLLARY 3. *Under the conditions of Theorem* 1, *if* $\omega_n(\delta)$ *is a modulus of continuity in I for the function* $f_n^{(p)}(x)$ (*assumed continuous*), *and* $\omega(\delta)$ *is the least modulus of continuity in I for* $f^{(p)}(x)$ (*assumed continuous*), *then we have for every* δ

$$\liminf_{n\to\infty} \omega_n(\delta) \geqq \omega(\delta).$$

The function $\omega(\delta)$ is said to be a *modulus of continuity* for the continuous function $\psi(x)$ in I if we have $|\psi(x+h)-\psi(x)| \leqq \omega(\delta)$ whenever x and $x+h$ lie in I, with $|h| \leqq \delta$, and if $\lim_{\delta\to 0} \omega(\delta)=0$. Under the hypothesis of Corollary 3 let δ and $\epsilon(>0)$ be arbitrary. There exist x and $x+h$ in I satisfying $|f^{(p)}(x+h)-f^{(p)}(x)| \geqq \omega(\delta)-\epsilon/3$, $|h|<\delta$. Choose $\delta_1(>0)$, $|h|+2\delta_1 \leqq \delta$. For n sufficiently large we have for some x_1 and x_2 (depending on n)

$$|f_n^{(p)}(x_1) - f^{(p)}(x)| < \epsilon/3, \qquad |x - x_1| < \delta_1,$$
$$|f_n^{(p)}(x_2) - f^{(p)}(x+h)| < \epsilon/3, \qquad |x + h - x_2| < \delta_1,$$

from which we may write (since $|x_1-x_2|<\delta$)

$$\omega_n(\delta) \geqq |f_n^{(p)}(x_1) - f_n^{(p)}(x_2)| \geqq \omega(\delta) - \epsilon,$$

whence the conclusion follows.

In this same circle of ideas we prove

THEOREM 2. *Suppose all the functions $f_n^{(p)}(x)$ are continuous and have the modulus of continuity $\omega(\delta)$ in the interval I: $a \leqq x \leqq b$, and suppose the sequence $f_n(x)$ converges to a function $f(x)$ on I. Then $f^{(p)}(x)$ exists on I and has there the modulus of continuity $\omega(\delta)$.*

The functions $f_n(x)$ are in fact equicontinuous on I, so it is sufficient to suppose the sequence $f_n(x)$ convergent to $f(x)$ on a set everywhere dense in I; indeed it is sufficient for the existence of $f(x)$ to assume the sequence $f_n(x)$ convergent in $p+1$ points of I. To prove Theorem 2 in the case $p>1$ we need two lemmas.

LEMMA 1. *If $f_n'(x)$ is continuous in I and becomes positively infinite there uniformly, then there is at most one point in I at which $f_n(x)$ converges or is bounded. If such a point ξ_0 exists, in any closed subinterval of I to the right of ξ_0 we have uniformly $f_n(x) \to +\infty$, and in any closed subinterval of I to the left of ξ_0 we have uniformly $f_n(x) \to -\infty$.*

If there exist two points ξ_0 and ξ_1 $(>\xi_0)$ in I at which $f_n(x)$ converges (or is bounded), we have

$$f_n(\xi_1) - f_n(\xi_0) = \int_{\xi_0}^{\xi_1} f_n'(x)dx \to +\infty,$$

which is impossible.

This last equation also shows that if $f_n(\xi_0)$ converges, then $f_n(\xi_1) \to +\infty$ for every ξ_1 in I with $\xi_1 > \xi_0$, and a similar equation shows that if $f_n(\xi_1)\to+\infty$ then $f_n(x)\to+\infty$ uniformly in I for $x\geqq\xi_1$. It may be proved similarly that if $f_n(\xi_0)$ converges, then at every x in I to the left of ξ_0 we have $f_n(x)\to-\infty$, uniformly on any closed subinterval of I to the left of ξ_0.

LEMMA 2. *If $f_n^{(p)}(x)$ is continuous in I and becomes positively infinite there uniformly, there are at most p points of I at which the sequence $f_n(x)$ converges or is bounded. If there are p such points of I, these points divide I into at most $p+1$ subintervals I_j; interior to each I_j we have $f_n(x)\to+\infty$ or $f_n(x)\to-\infty$, uniformly on any closed subinterval interior to I_j.*

Lemma 2 is a consequence of Lemma 1, by application of Lemma 1 to the subintervals of I in which $f_n^{(p-1)}(x)$ becomes positively and negatively infinite, respectively. Under the hypothesis of Lemma 2 with $p=2$, suppose the sequence $f_n'(x)$ convergent or even bounded in a point ξ_0 interior to I; it cannot occur that $f_n(x)$ should converge at ξ_0 as well as at a point $\xi_1(>\xi_0)$ of I and at a point ξ_2 $(<\xi_0)$ of I, for under those conditions by Lemma 1 we should have $f_n(x)\to+\infty$ or $f_n(x)\to-\infty$ uniformly in a subinterval of each of the intervals $\xi_2<x<\xi_0$, $\xi_0<x<\xi_1$; each of the latter intervals contains for n sufficiently large a maximum or minimum of $f_n(x)$ and thus at least one zero of $f_n'(x)$, so for n sufficiently large I contains at least one zero of $f_n''(x)$, in contradiction to our hypothesis. Indeed it follows from this same reasoning applied to a suitable subsequence that $(p=2)$ if the sequence $f_n(x)$ converges in each of two points η_1 and η_2 $(<\eta_1)$ of I, then in each of the subintervals of I that exist: $a\leqq x<\eta_2$, $\eta_2<x<\eta_1$, $\eta_1<x\leqq b$ we have $f_n(x)\to+\infty$ or $f_n(x)\to-\infty$ and uniformly in each closed subsubinterval. Continued application of this argument establishes Lemma 2.

The number $p+1$ of subintervals I_j of Lemma 2 may actually be attained, as is shown by the example

$$ {}_n(x) \equiv n\left(x-\frac{1}{p+1}\right)\left(x-\frac{2}{p+1}\right)\cdots\left(x-\frac{p}{p+1}\right), \qquad I:\ 0\leqq x\leqq 1, $$

a polynomial of degree p, whose pth derivative becomes positively infinite. The relations $f_n(x)\to+\infty$ and $f_n(x)\to-\infty$ of Lemma 2 do not necessarily hold in alternate intervals I_j, as we see from the example $f_n(x)\equiv n(x-1/2)^2$, I: $0\leqq x\leqq 1$, with $p=2$; we have $f_n^{(p)}(x)\equiv 2n\to+\infty$.

We return to the proof of Theorem 2. The functions $f_n^{(p)}(x)$ are equicontinuous on I, so every subsequence which is bounded in a point of I is uniformly bounded in I; any subsequence which becomes positively (or negatively) infinite in a point of I becomes positively (or negatively) infinite uniformly in I. From Lemma 2 it follows that no subsequence can become positively or negatively infinite uniformly in I, so the functions $f_n^{(p)}(x)$ are uniformly bounded in I. The functions $f_n^{(p-1)}(x)$ have their first derivatives uniformly bounded in I, hence are equicontinuous in I. By the argument just given for the functions $f_n^{(p)}(x)$, and by application of Lemma 2, it follows that the set $f_n^{(p-1)}(x)$ is uniformly bounded in I, and by continuing this argument it follows that each of the sets $f_n^{(p-2)}(x), \cdots, f_n'(x), f_n(x)$ is uniformly bounded and equicontinuous in I. Then for a suitably chosen sequence of integers n_k, it is true that at a set of points everywhere dense in I, each of the sequences $f_{n_k}(x), f_{n_k}'(x), \cdots, f_{n_k}^{(p)}(x)$ converges, hence converges uniformly in I to some limit function; we denote these limit functions by $F_0(x), F_1(x), \cdots, F_p(x)$. From the hypothesis of Theorem 2 we have $F_0(x) \equiv f(x)$. From the uniformity of the convergence of the sequence $f_{n_k}'(x)$ it follows by the classical theorem on term-by-term differentiation of series that $f'(x)$ exists and we have $F_1(x) \equiv f'(x)$. Repetition of this reasoning shows that $f''(x), \cdots, f^{(p)}(x)$ all exist and we have $F_2(x) \equiv f''(x), \cdots, F_p(x) \equiv f^{(p)}(x)$. The remainder of Theorem 2 follows from Corollary 3 to Theorem 1.

Under the hypothesis of Theorem 2, we have essentially shown that every subsequence of the functions $f_n^{(p)}(x)$ contains a new subsequence which converges uniformly in I to $f^{(p)}(x)$, from which it follows that the sequence $f_n^{(p)}(x)$ itself converges uniformly in I to $f^{(p)}(x)$, and that the sequences $\{f_n^{(p-1)}(x)\}, \{f_n^{(p-2)}(x)\}, \cdots, \{f_n(x)\}$ converge uniformly in I to the respective limits $f^{(p-1)}(x), f^{(p-2)}(x), \cdots, f(x)$.

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