

KERNEL FUNCTIONS AND CLASS L^2

I. EDWARD BLOCK¹

1. Introduction. Let $f(z)$ be a measurable function of the real variables x and y in the plane of the complex $z(=x+iy)$ variable. Denote by B a finite domain bounded by a finite number of analytic Jordan curves, and by $L^2(B)$ the Hilbert space of functions $f(z)$ of summable square on B , i.e.,

$$\|f\|_B^2 = \iint_B dx dy |f(z)|^2.$$

The subspace of functions of $L^2(B)$ which are analytic on B is designated by $\mathcal{L}^2(B)$. Associated with B is the classical reproducing kernel $K(z, \zeta^*)$ [1], and it is well known that if $f_1(z)$ belongs to $\mathcal{L}^2(B)$, then

$$(1.1) \quad f_1(z) = \iint_B d\xi d\eta K(z, \zeta^*) f_1(\zeta) \quad (\zeta = \xi + i\eta).$$

(* denotes the complex conjugate.) Also there is a complete orthonormal set of functions on B which span $\mathcal{L}^2(B)$ and for which there exists a Riesz-Fischer theory with convergence in L^2 norm. It follows immediately (Walsh [3, pp. 149–151]; Bergman [1, pp. 5–10]) that (1.1) may be applied to functions $f(z)$ in $L^2(B)$, and the result is the projection $f_1(z)$ of $f(z)$ on $\mathcal{L}^2(B)$. That is,

$$f_1(z) = \iint_B d\xi d\eta f(\zeta) K(z, \zeta^*) \quad (\text{for } z \text{ in } B).$$

Recently, Bergman and Schiffer [2] have introduced to the theory of kernel functions the new kernel

$$L(z, \zeta) = -\frac{2}{\pi} \frac{\partial^2 g(z, \zeta)}{\partial z \partial \zeta}.$$

$g(z, \zeta)$ is the Green's function for B , and

$$\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right).$$

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It is shown that

$$L(z, \zeta) = \frac{1}{\pi(z - \zeta)^2} - l(z, \zeta),$$

where $l(z, \zeta)$ is a single-valued analytic function in both variables on the closure of B . In addition, if $f_1(z)$ is in $\mathcal{L}^2(B)$ and if

$$P. \int \int_B d\xi d\eta L(z, \zeta) [f_1(\zeta)]^* = \lim_{\epsilon \rightarrow 0} \int \int_{B: |z-\zeta| > \epsilon} d\xi d\eta L(z, \zeta) [f_1(\zeta)]^*,$$

then Bergman and Schiffer have proved that

$$(1.2) \quad P. \int \int_B d\xi d\eta L(z, \zeta) [f_1(\zeta)]^* = 0.$$

Next let $f_2(z)$ be determined by the relation

$$(1.3) \quad f(z) = f_1(z) + f_2(z).$$

Then $f_2(z)$ belongs to the subspace which is the orthogonal complement to $\mathcal{L}^2(B)$. Indeed,

$$\int \int_B d\xi d\eta f_1(\zeta) [f_2(\zeta)]^* = 0.$$

It is the purpose of this paper to study the relationship of the L -kernel to $L^2(B)$. Just as the K -kernel determines the projection $f_1(z)$ of $f(z)$ on $\mathcal{L}^2(B)$, so the L -kernel determines the projection $f_2(z)$ of $f(z)$ on the orthogonal complement to $\mathcal{L}^2(B)$. In fact, if we define $f(z) = 0$ exterior to B and take

$$L(f) = \text{l.i.m.}_{\epsilon \rightarrow 0} \int \int_{B: |\zeta-z| > \epsilon} d\xi d\eta L(z, \zeta) [f(\zeta)]^*,$$

then $L[L(f)] = f_2(z)$ almost everywhere in B . In addition, it is found that the subspace of functions which satisfy a Lipschitz condition of order α , $0 < \alpha < 1$, on B is invariant under the L -transformation.

2. A theorem of Beurling. In private conversation with this author, Beurling has communicated the following result.

THEOREM. *If (a) $f(z) \in L^2$ (over the complex plane),*

$$(b) \quad g_\epsilon(z) = \frac{1}{\pi} \int \int_{|\zeta-z| > \epsilon} d\xi d\eta \frac{f(\zeta)}{(\zeta - z)^2},$$

then there exists a function $g(z)$ in L^2 such that

- (a) $g(z) = T(f) = \text{l.i.m.}_{\epsilon \rightarrow 0} \frac{1}{\pi} \iint_{|\zeta-z|>\epsilon} d\xi d\eta \frac{f(\zeta)}{(\zeta - z)^2},$
- (b) $\{T[T(f)]^*\}^* = T[T(f^*)]^* = f(z) \text{ a. e.},$
- (c) $\iint dxdy |f(z)|^2 = \iint dxdy |g(z)|^2.$

Unless otherwise indicated, the range of integration here and in what follows will be the entire complex plane. To prove the theorem, one first shows the existence of the Fourier-Plancherel transform of $g(z)$. Indeed, the two-dimensional transform of $g(z)$ is $(-z^2/2\pi|z|^2)F(x, y)$ where $(1/2\pi)F(x, y)$ is the transform of $f(\zeta)$. Application of this to the inversion formulae in the statement of the theorem yields the proof.

3. $L(z, \zeta)$ and the class $L^2(B)$. Let $f(z)$ be any function of $L^2(B)$. We define $f(z) = 0$ exterior to B . It is evident from the Schwarz inequality that $\iint_B d\xi d\eta l(z, \zeta) [f(\zeta)]^*$ converges absolutely. From this and Beurling's theorem, one now finds that a function $g_2(z)$ of summable square on B exists such that

$$(3.1) \quad \lim_{\epsilon \rightarrow 0} \left\| g_2(z) - \frac{1}{\pi} \iint_{B: |\zeta-z|>\epsilon} d\xi d\eta L(z, \zeta) [f(\zeta)]^* \right\|_B = 0.$$

Now $f_2(z)$, as defined by (1.3), is orthogonal to each member of $L^2(B)$. Therefore we may conclude that

$$(3.2) \quad \iint_B d\xi d\eta l(z, \zeta) [f_2(\zeta)]^* = 0$$

for each z in B . It now follows from (1.2), (3.1) and (3.2) that

$$(3.3) \quad \lim_{\epsilon \rightarrow 0} \left\| g_2(z) - \frac{1}{\pi} \iint_{B: |\zeta-z|>\epsilon} d\xi d\eta \frac{[f_2(\zeta)]^*}{(\zeta - z)^2} \right\|_B = 0.$$

Accordingly, it is consistent to define $g_2(z)$ in the entire complex plane by means of (3.3), for it reduces to (3.1) when z is in B . When z is exterior to B , $g_2(z)$ vanishes since the kernel becomes an analytic function of ζ on B . Next we prove a

LEMMA. *If z is in B , then*

$$\iint_B d\xi d\eta K(z, \zeta^*) g_2(\zeta) = 0.$$

For each positive ϵ , ($w = u + iv$),

$$\iint_{B:|\zeta-w|>\epsilon} dudv \frac{[f_2(w)]^*}{(\zeta-w)^2}$$

is of summable square on B , as is $K(z, \zeta^*)$ with respect to ζ for each fixed z in B . We replace $g_2(z)$ in the lemma by its value (3.3). Then it follows from a well known theorem for sequences of functions convergent in the mean that

$$\begin{aligned} \iint_B d\xi d\eta K(z, \zeta^*) \frac{1}{\pi} \text{l.i.m.}_{\epsilon \rightarrow 0} \iint_{B:|\zeta-w|>\epsilon} dudv \frac{[f_2(w)]^*}{(\zeta-w)^2} \\ = \lim_{\epsilon \rightarrow 0} \iint_B d\xi d\eta K(z, \zeta^*) \frac{1}{\pi} \iint_{B:|\zeta-w|>\epsilon} dudv \frac{[f_2(w)]^*}{(\zeta-w)^2}. \end{aligned}$$

The integrals on the right are absolutely convergent for each z in B , so we may interchange the order of integration. This produces

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \iint_B dudv [f_2(w)]^* \frac{1}{\pi} \iint_{B:|w-\zeta|>\epsilon} d\xi d\eta \frac{K(z, \zeta^*)}{(w-\zeta)^2} \\ (3.4) \quad = \iint_B dudv [f_2(w)]^* \frac{1}{\pi} \text{l.i.m.}_{\epsilon \rightarrow 0} \iint_{B:|w-\zeta|>\epsilon} d\xi d\eta \frac{K(z, \zeta^*)}{(w-\zeta)^2}. \end{aligned}$$

It is a result of Bergman and Schiffer [2, p. 216] that

$$l(z, w) = \frac{1}{\pi} P. \int \int_B d\xi d\eta [K(\zeta, w^*)]^* \frac{1}{(\zeta-z)^2},$$

and the existence of this principal value implies that it is the same as the result obtained by taking the l.i.m. Inasmuch as $[K(\zeta, w^*)]^* = K(w, \zeta^*)$, it now follows that the right member of (3.4) is

$$\iint_B dudv [f_2(w)]^* l(z, w).$$

Since this is zero, the lemma is proved.

We may now state

THEOREM 3.1. *If $f(z) \in L^2(B)$, then*

$$L[L(f)] = f_2(z) \quad \text{a.e. in } B.$$

If we define $f_2(z) = 0$ for z exterior to B , then we may write $g_2(z) = T(f_2^*)$. It follows from Beurling's theorem that $T(g_2^*) = f_2(z)$. Since $g_2(z) = 0$ for z exterior to B , we have

$$T(g_2^*) = \text{l.i.m.}_{\epsilon \rightarrow 0} \frac{1}{\pi} \iint_{B:|z-\zeta|>\epsilon} d\xi d\eta \frac{[g_2(\zeta)]^*}{(z-\zeta)^2}.$$

But the lemma implies that $g_2(z)$ belongs to the orthogonal complement to the space $\mathcal{L}^2(B)$. Therefore

$$L(g_2) = - \iint_B d\xi d\eta l(z, \zeta) [g_2(\zeta)]^* + \text{l.i.m.}_{\epsilon \rightarrow 0} \frac{1}{\pi} \iint_{B: |z-\zeta| > \epsilon} d\xi d\eta \frac{[g_2(\zeta)]^*}{(z - \zeta)^2} \\ = T(g_2^*) = f_2(z).$$

Since $L(g_2) = L[L(f)]$, the theorem follows.

COROLLARY. $\|f_2\|_B = \|g_2\|_B$.

COROLLARY. If $f(z) \in L^2(B)$, then $L[L(f)] = 0$ a.e. in B if and only if $f(z) \in \mathcal{L}^2(B)$.

4. Invariance of the class of functions Lip α on B . We first consider the class of functions which satisfy a Lipschitz condition uniformly over the complex plane.

- THEOREM 4.1.** If (a) $\iint dx dy |f(z)|^2 < \infty$,
 (b) $|f(z_1) - f(z_2)| \leq A |z_1 - z_2|^\alpha$ when $|z_1 - z_2| < \delta$,
 (c) $0 < \alpha < 1$,

then $g(z) (= T(f))$ satisfies a Lipschitz condition of order α .

Under the conditions stated, the principal value integral

$$\frac{1}{\pi} P. \iint d\xi d\eta \frac{f(\zeta)}{(\zeta - z)^2}$$

exists. Inasmuch as

$$\frac{1}{\pi} P. \iint_{|\zeta-z| < \delta} d\xi d\eta \frac{1}{(\zeta - z)^2} = 0,$$

it follows that

$$\frac{1}{\pi} P. \iint d\xi d\eta \frac{f(\zeta)}{(\zeta - z)^2} = \frac{1}{\pi} \iint_{|\zeta-z| > \delta} d\xi d\eta \frac{f(\zeta)}{(\zeta - z)^2} \\ + \frac{1}{\pi} \iint_{|\zeta-z| < \delta} d\xi d\eta \frac{f(\zeta) - f(z)}{(\zeta - z)^2}.$$

We note that the last integral on the right is absolutely convergent because of the Lipschitz condition on $f(z)$. The first integral on the right also converges, and thus the principal value integral exists.

It is also evident that the conditions of the theorem imply that $f(z)$ is continuous and bounded in the z -plane, tending uniformly to zero as $|z| \rightarrow \infty$.

Now consider any two points z_1 and z_2 such that $|z_1 - z_2| < \delta$. A short calculation yields

$$(4.1) \quad g(z_1) - g(z_2) = \frac{2(z_1 - z_2)}{\pi} P. \iint d\xi d\eta [f(\zeta) - f(z_1)] \frac{\zeta - (z_1 + z_2)/2}{(\zeta - z_1)^2(\zeta - z_2)^2}.$$

Here $P.$ indicates that the principal value is to be calculated with respect to both singularities of the integrand. We observe that the region $C_\delta: |\zeta - z_2| \leq 2\delta$ contains $C_{12}: |\zeta - z_2| \leq 2|z_1 - z_2|$, and we denote by M and N the regions given respectively by $M: \{|\zeta - z_2| \leq 2\delta, |\zeta - z_2| > 2|z_1 - z_2|\}$ and $N: \{|\zeta - z_2| \leq 2|z_1 - z_2|\}$. If ζ belongs to M , then it may be verified from the geometry that

$$(4.2) \quad \frac{1}{2} \leq \left| \frac{\zeta - z_2}{\zeta - z_1} \right| \leq 2, \quad \frac{1}{2} \leq \left| \frac{\zeta - (z_1 + z_2)/2}{\zeta - z_2} \right| \leq 2.$$

Designate by I, II, and III respectively those parts of the right side of (4.1) taken over the regions M, N , and $\bar{C}_\delta: |\zeta - z_2| > 2\delta$. Using the inequalities (4.2) and the Lipschitz condition for $f(z)$, the integral I is in absolute value less than or equal to

$$\begin{aligned} \frac{2^{4+\alpha}A |z_1 - z_2|}{\pi} \iint_M d\xi d\eta |\zeta - z_2|^{\alpha-3} &= \frac{2^{4+2\alpha}A}{1-\alpha} [|z_1 - z_2|^\alpha - \delta^{\alpha-1} |z_1 - z_2|] \\ &= O(|z_1 - z_2|^\alpha). \end{aligned}$$

We rewrite the integral II in the form

$$\frac{1}{\pi} P. \iint_N d\xi d\eta \left[\frac{f(\zeta) - f(z_1)}{(\zeta - z_1)^2} - \frac{f(\zeta) - f(z_2)}{(\zeta - z_2)^2} + \frac{f(z_2) - f(z_1)}{(\zeta - z_2)^2} \right].$$

The third of these integrals is zero, and the first two converge absolutely in accordance with the Lipschitz condition for $f(z)$. An estimate for each of the first two integrals is readily obtained. We have

$$\begin{aligned} \left| \iint_N d\xi d\eta \frac{f(\zeta) - f(z_1)}{(\zeta - z_1)^2} \right| &\leq A \iint_{|\zeta - z_1| \leq 3|z_1 - z_2|} d\xi d\eta |\zeta - z_1|^{\alpha-2} \\ &= \frac{2\pi A}{\alpha} 3^\alpha |z_1 - z_2|^\alpha \\ &= O(|z_1 - z_2|^\alpha). \end{aligned}$$

The estimate is uniform for all z_1 and z_2 for which $|z_1 - z_2| < \delta$. The same estimate may be obtained for the second integral.

If ζ is in \bar{C}_δ , the inequalities (4.2) are still valid. Using these, we may estimate the integral III. Indeed, in absolute value, it is less than or equal to

$$\begin{aligned} & \frac{16|z_1 - z_2|}{\pi} \iint_{\bar{C}_\delta} d\xi d\eta \frac{|f(\zeta)|}{|\zeta - z_2|^3} \\ & \leq \frac{16|z_1 - z_2|}{\pi} \left[\iint d\xi d\eta |f(\zeta)|^2 \iint_{\bar{C}_\delta} d\xi d\eta \frac{1}{|\zeta - z_2|^6} \right]^{1/2} \\ & = O(|z_1 - z_2|) \end{aligned}$$

uniformly for all z_1 and z_2 for which $|z_1 - z_2| < \delta$. Thus it follows that (4.1) is uniformly $O(|z_1 - z_2|^\alpha)$, and the theorem is proved.

It is important to note that the Lipschitz condition on $f(z)$ need not hold uniformly. For example, if $f(z)$ satisfies a Lipschitz condition of order α at z_0 , then $g(z)$ satisfies the same order Lipschitz condition at z_0 . With this remark, we have immediately:

THEOREM 4.2. *Let $f(z)$ belong to $L^2(B)$ and satisfy a Lipschitz condition of order α at a point z_0 in B . Then $L(f)$ satisfies the same order Lipschitz condition at z_0 . Moreover, if $f(z)$ satisfies a Lipschitz condition of order α uniformly on B , so does $L(f)$.*

It follows from the previous remarks that the part of $L(f)$ which results from the singular kernel satisfies a Lipschitz condition of order α . The part which comes from $l(z, \zeta)$ satisfies a Lipschitz condition of order 1 because of the analyticity of $l(z, \zeta)$.

5. $L(z, \zeta)$ and the unit circle. For the unit circle,

$$L(z, \zeta) = \frac{1}{\pi(z - \zeta)^2}, \quad K(z, \zeta^*) = \frac{1}{\pi(1 - \zeta^*z)^2}.$$

While the general results already obtained easily reduce to the special case of the unit circle, the simplicity of $L(z, \zeta)$ leads to special results which may be of some interest.

Let $f(z)$ be a complex-valued function on $B: |z| < 1$ such that

$$\iint_{|z| < 1} dx dy |f(z)|^2 < \infty.$$

Define $f(z)$ and its two components $f_1(z)$ and $f_2(z)$, as given by (1.3), to be zero for $|z| > 1$. We consider the transformation

$$K(f) = \text{l.i.m.}_{\epsilon \rightarrow 0} \frac{1}{\pi} \iint_{|\zeta - 1/z^*| > \epsilon} d\xi d\eta \frac{f(\zeta)}{(1 - \zeta^*z)^2}.$$

When $|z| < 1$, it follows immediately from the classical kernel theory that $K(f) = f_1(z)$. On the other hand, if $|z| > 1$, we have $K(f) = K(f_1) + K(f_2)$. Application of Green's theorem with respect to the region $B_\epsilon: \epsilon < |\zeta - 1/z^*| < 1$ shows that $K(f_1) = 0$, and from (3.3), we find that $K(f_2) = 1/z^2 [g_2(1/z^*)]^*$. Then we have

$$K(f) = \frac{1}{z^2} \left[g_2 \left(\frac{1}{z^*} \right) \right]^*, \quad (\text{for } |z| > 1).$$

To compute $K^2(f)$, we first observe that $g_2(1/z) = 0$ for $|z| < 1$, and $f_1(z) = 0$ for $|z| > 1$. Then

$$K^2(f) = f_1(z) + K \left[\frac{1}{z^2} \left[g_2 \left(\frac{1}{z^*} \right) \right]^* \right] \quad \text{a.e.}$$

The last member on the right side is

$$\begin{aligned} \text{l.i.m.}_{\epsilon \rightarrow 0} \frac{1}{\pi} \iint_{|\zeta - 1/z^*| > \epsilon} d\xi d\eta \frac{1/\zeta^2 [g_2(1/\zeta^*)]^*}{(1 - \zeta^*z)^2} \\ = \text{l.i.m.}_{\epsilon \rightarrow 0} \frac{1}{\pi} \iint_{|(w-z)/wz| > \epsilon} dudv \frac{[g_2(w)]^*}{(w-z)^2} \end{aligned}$$

since the Jacobian of the transformation $\zeta = 1/w^*$ is $1/|w|^4$. It can be shown that this last integral is equal to

$$\text{l.i.m.}_{\epsilon \rightarrow 0} \frac{1}{\pi} \iint_{|w-z| > \epsilon} dudv \frac{[g_2(w)]^*}{(w-z)^2},$$

whence it follows that $K^2(f) = f_1(z) + [T^*(g_2)]^* = f_1(z) + f_2(z)$ a.e. Then our result is that

$$K^2(f) = f(z) \quad \text{a.e.}$$

BIBLIOGRAPHY

1. S. Bergman, *The kernel function and conformal mapping*, Mathematical Surveys, No. 5, New York, 1950.
2. S. Bergman and M. Schiffer, *Kernel functions and conformal mapping*, *Compositio Math.* vol. 8 (1951) pp. 205-249.
3. J. L. Walsh, *Interpolation and approximation by rational functions in the complex domain*, Amer. Math. Soc. Colloquium Publications, vol. 20, New York, 1950.

HARVARD UNIVERSITY AND
PHILCO CORPORATION