AN INEQUALITY FOR MINKOWSKI MATRICES[1]

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Introduction. The class of Minkowski matrices consists of square matrices of the form $\delta - a$, where δ is the identity matrix and a, with real or complex elements, satisfies the condition (1). The inequality is given in the lemma, which improves the author's previous result [**3**, p. 239] by removing two restrictions.[2] Refinements of the inequality are given in §3.

G. B. Price [**2**] and A. M. Ostrowski [**1**] give bounds for determinants with dominant principal diagonal. It loses no generality to consider the square matrices with units on the principal diagonal. Thus our results may be applied to the determinants studied by Price and Ostrowski. We apply the inequality of our lemma to obtain bounds for the determinant of $\delta - a$. Our results in (9) and (15) are better estimates than those of Price and Ostrowski. The main idea of our method centers on (13) and (14). The concept of quasi-inverse, which was used in [**3**], is no longer needed.

We use the notation $a(i, j)$ instead of a_{ij}.

1. An inequality for Minkowski matrices. We assume that

$$(1) \qquad s^{(n)}(j) = \sum_{i=1}^{n} | a(i, j) | \leqq 1, \quad | a(j, j) | < 1 \qquad (j = 1, \cdots, n).$$

The notation $s^{(k)}(j)$, $j = 1, \cdots, k$, has similar meaning.

In the sequel, we let $\delta - a_k$ be the principal minor, which consists of the first k rows and columns of $\delta - a$. δ will always be the identity matrix of the same order as a_k. Let M_k and D_k be the adjoint and determinant of $\delta - a_k$ respectively. For simplicity, we let all the summations extend from 1 to $n-1$, unless otherwise specified.

LEMMA. *If* (1) *is satisfied for* $j = 1, \cdots, n-1$, *then*

$$(2) \quad \sum_{i} | a(n, i) | \cdot | M_{n-1}(i, k) | \leqq s^{(n)}(k) | D_{n-1} |, \quad k = 1, \cdots, n-1.$$

PROOF. From (1), we have

Received by the editors April 28, 1952.

[1] The paper was prepared under the Navy contract N6onr-27009.

[2] In [**3**], the author considered only matrices with non-negative elements, and assumed the nonsingularity for the principal minor of $\delta - a$ with the nth row and column omitted.

$$|a(n, j)| = s^{(n)}(j) - \sum_i |a_{n-1}(i, j)| \qquad (j = 1, \cdots, n-1).$$

Multiplying both sides by $|M_{n-1}(j, k)|$ and summing for $j = 1, \cdots, n-1$, we get

$$\begin{aligned} \sum_j |a(n, j)| \cdot |M_{n-1}(j, k)| &= \sum_j s^{(n)}(j) |M_{n-1}(j, k)| \\ &\quad - \sum_j \sum_i |a_{n-1}(i, j)| \cdot |M_{n-1}(j, k)|. \end{aligned} \tag{3}$$

By putting

$$|M_{n-1}(j, k)| = |D_{n-1}| \delta(j, k) + |M_{n-1}(j, k)| - |D_{n-1}| \delta(j, k), \tag{3.5}$$

the first summation on the right side of (3) becomes

$$\begin{aligned} &|D_{n-1}| s^{(n)}(k) + \sum_j s^{(n)}(j) [|M_{n-1}(j, k)| - |D_{n-1}| \delta(j, k)] \\ &\qquad \leqq |D_{n-1}| s^{(n)}(k) + \sum_j s^{(n)}(j) |M_{n-1}(j, k) - D_{n-1}\delta(j, k)|. \end{aligned} \tag{4}$$

The double summation on the right-hand side of (3) is

$$\begin{aligned} \sum_i \sum_j |a_{n-1}(i, j) M_{n-1}(j, k)| &\geqq \sum_i \left| \sum_j a_{n-1}(i, j) M_{n-1}(j, k) \right| \\ &= \sum_i |M_{n-1}(i, k) - D_{n-1}\delta(i, k)|, \end{aligned} \tag{5}$$

since $(\delta - a_{n-1}) M_{n-1} = D_{n-1}\delta$. Combining (3), (4), and (5), we have

$$\begin{aligned} &\sum_j |a(n, j)| \cdot |M_{n-1}(j, k)| \\ &\leqq |D_{n-1}| s^{(n)}(k) - \sum_j (1 - s^{(n)}(j)) |M_{n-1}(j, k) - D_{n-1}\delta(j, k)| \\ &\leqq |D_{n-1}| s^{(n)}(k), \end{aligned} \tag{6}$$

which proves our lemma.

Let $\delta - a_{n-1}$ be nonsingular and R_{n-1} be its inverse. Then, with assumption (1),

$$\sum_j |a(n, j) R_{n-1}(j, k)| \leqq s^{(n)}(k), \qquad k = 1, \cdots, n-1. \tag{7}$$

2. **An application.** By Cauchy's expansion,

$$D_n = [1 - a(n, n)] D_{n-1} - \sum_{j,k} a(n, j) M_{n-1}(j, k) a(k, n), \tag{8}$$

where M_{n-1} is the adjoint of $\delta - a_{n-1}$. *With assumption* (1), *it follows from* (2) *and* (8) *that*

$$\begin{aligned}(9)\qquad [|1-a(n, n)|-N^{(n)}]|D_{n-1}| \\ \leqq |D_n| \leqq [|1-a(n, n)|+N^{(n)}]|D_{n-1}|,\end{aligned}$$

where

$$(10)\qquad N^{(n)} = \sum_{j=1}^{n-1} s^{(n)}(j)\,|a(j, n)|.$$

It requires to show that $|1-a(n, n)|-N^{(n)}$ is non-negative, for otherwise the lower bound would not be effective. By (10) and (1),

$$\begin{aligned}(11)\qquad N^{(n)} &= \sum_j s^{(n)}(j)\,|a(j, n)| \\ &\leqq \sum_j |a(j, n)| \leqq 1-|a(n, n)| \leqq |1-a(n, n)|.\end{aligned}$$

This completes the proof of (9).

COROLLARY 1. *If condition* (1) *holds, and for* $k=2, \cdots, n$,

$$\sum_{j=1}^{k-1} s^{(k)}(j)\,|a(j, k)| < |1-a(k, k)|, \textit{ then } D_n \neq 0.$$

COROLLARY 2. *If condition* (1) *holds, then*

$$(12)\qquad \sum_{i,j} |a(n, i)M_{n-1}(i, j)a(j, n)| \leqq [1-|a(n, n)|]\,|D_{n-1}|.$$

Inequality (12) follows from (11), (2), and (8).

A better approximation for D_n may be obtained as follows: Let $k=2, \cdots, n$, and M_k be the adjoint of $\delta-a_k$. Then

$$(13)\qquad M_k - D_k\delta = a_kM_k = M_ka_k.$$

Repeated applications of (13) give

$$(14)\qquad M_k = (\delta + a_k + \cdots + \overset{m}{a_k})D_k + M_k\overset{m+1}{a_k}, \quad \overset{m}{a_k} = (a_k)^m, \quad m \geqq 0.$$

Our assumption is given by (1). Substituting (14) with $k=n-1$ into (8) and making use of (2), we have

$$(15)\qquad (|S_m^{(n)}|-P_m^{(n)})\,|D_{n-1}| \leqq |D_n| \leqq (|S_m^{(n)}|+P_m^{(n)})\,|D_{n-1}|, \quad m \geqq 0,$$

where

$$(16)\qquad S_m^{(n)} = 1-a(n, n) - \sum_{i,j<n} a(n, i)\sum_{k=0}^{m} \overset{k}{a}_{n-1}(i, j)a(j, n), \quad m \geqq 0,$$

$$P_m^{(n)} = \sum_i s^{(n)}(i) \left| \sum_j a_{n-1}^{m+1}(i, j) a(j, n) \right|, \qquad m \geqq 0. \tag{17}$$

We may replace $P_m^{(n)}$ by

$$Q_m^{(n)} = c^{(n)} \sum_i s^{(n-1)}(i) \left| \sum_j a_{n-1}^{m}(i, j) a(j, n) \right|, \qquad m \geqq 0, \tag{18}$$

where

$$c^{(n)} = \max \left[s^{(n)}(j) \text{ for } j = 1, 2, \cdots, n-1 \right]. \tag{19}$$

Obviously,

$$c^{(n)} \leqq 1, \qquad P_m^{(n)} \leqq Q_m^{(n)}, \qquad m \geqq 0. \tag{20}$$

To show that $|S_m^{(n)}| - Q_m^{(n)}$ is non-negative, observe that

$$|S_0^{(n)}| - Q_0^{(n)} \geqq |1 - a(n, n)| - N^{(n)} \geqq 0, \tag{21}$$

$$|S_{r+1}^{(n)}| - Q_{r+1}^{(n)} \geqq |S_r^{(n)}| - P_r^{(n)} \geqq |S_r^{(n)}| - Q_r^{(n)}. \tag{22}$$

The proof may, thus, be completed by induction. One can easily furnish the detail by using the triangle property of the complex numbers.

3. **Refinements of inequality (2).** *Condition* (1) *is assumed for* $j = 1, \cdots, n-1$. The interesting part is that we may use inequality (2) to obtain some refinement. The first method has its underlying idea given in (13) and (14). By (3.5),

$$\begin{aligned} \sum |a(n, j)| \, (|D_{n-1}| \, \delta(j, k) + |M_{n-1}(j, k)| - |D_{n-1}| \, \delta(j, k)) & \\ \leqq |a(n, k)| \cdot |D_{n-1}| & \\ + \sum |a(n, j)| \cdot |M_{n-1}(j, k) - D_{n-1}\delta(j, k)| & \\ \leqq |a(n, k)| \cdot |D_{n-1}| & \\ + \sum |a(n, j)| \cdot \left| \sum M_{n-1}(j, h) a_{n-1}(h, k) \right| & \\ \leqq |a(n, k)| \cdot |D_{n-1}| & \\ + \sum \left(\sum |a(n, j) M_{n-1}(j, h)| \right) |a_{n-1}(h, k)| & \\ \leqq \left(|a(n, k)| + \sum s^{(n)}(h) \, |a_{n-1}(h, k)| \right) |D_{n-1}|. & \end{aligned} \tag{23}$$

The quantity within the parentheses is less than $s^{(n)}(k)$ if $s^{(n)}(j) < 1$ for some $j = 1, \cdots, n-1$. If we use (23) in (8), the resulting inequalities are better than (9), but not as good as (15) for $m = 0$.

The second method is as follows: Let d be a diagonal matrix with

$1/(1-a(k,k))$ on the kth row and column. Then $(\delta-a)d=\delta-a^0$, where $a^0(i,i)=0$ and $a^0(i,k)=a(i,k)/(1-a(k,k))$ for $i\neq k$. Applying (2) to $\delta-a^0$, we obtain, after simplification,

$$(24)\qquad \sum |a(n,i)|\cdot|M_{n-1}(i,k)| \leqq |D_{n-1}| \frac{s^{(n)}(k)-|a(k,k)|}{|1-a(k,k)|},$$

which is strictly less than $s^{(n)}(k)|D_{n-1}|$ if $a(k,k)\neq 0$ and $s^{(n)}(k)<1$ (and in that case the inequality (24) is also strictly $<$). The preceding inequality is sharper than the result of Ostrowski.[3]

References

1. A. M. Ostrowski, *Note on bounds for determinants with dominant principal diagonal*, Proceedings of American Mathematical Society vol. 3 (1952) pp. 26–30.

2. G. B. Price, *Bounds for determinants with dominant principal diagonal*, Proceedings of American Mathematical Society vol. 2 (1951) pp. 497–502.

3. Y. K. Wong, *Some inequalities for determinants of Minkowski type*, Duke Math. J. vol. 19 (1952) pp. 231–241.

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[3] See [1, inequality (13)]. Ostrowski actually proved that $|M(n,k)/D_n| \leqq |D_{n-1}/D_n|\sigma_k$ ($M=\mathrm{adj}(\delta-a)$, $D_{n-1}=M(n,n)$) where σ_k is the fractional expression on the right side of (24), under the assumption that all $\sigma_k<1$ to assure the nonvanishing of D_n. Note that $\pm M(n,k)=\sum a(n,i)M_{n-1}(i,k)$.