ON THE CONVERGENCE-ABSCISSAS OF THE GENERALIZED FACTORIAL SERIES

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1. Introduction. We consider the generalized factorial series

$$F(s) = \sum_{n=1}^{\infty} a_n[\lambda_1\lambda_2 \cdots \lambda_n][(s + \lambda_1)(s + \lambda_2) \cdots (s + \lambda_n)]^{-1}, \tag{1.1}$$

$$s = \sigma + it, \lambda_n = r_n e^{i\phi_n} \ (n = 1, 2, \cdots),$$

where

$$\lim_{n\to\infty} r_n = + \infty, \ |\phi_n| \leqq \phi < \pi/2 \qquad (n = 1, 2, \cdots). \tag{1.2}$$

In his classical note [1, §6], E. Landau has studied (1.1) in the case in which $\sum_{n=1}^{\infty} 1/r_n = +\infty$, $\phi_n = 0$ $(n=1, 2, \cdots)$. Under additional conditions, he has determined convergence-abscissas of (1.1) in terms of coefficients a_n $(n=1, 2, \cdots)$. S. Pincherle [2], G. Belardinelli [3], and T. Fort [4, 5] have studied (1.1) with complex λ_n $(n=1, 2, \cdots)$ satisfying (1.2) and some other conditions. In this note, without any additional conditions, we shall determine the convergence-abscissas of (1.1) with real λ_n $(n=1, 2, \cdots)$ in terms of coefficients a_n $(n=1, 2, \cdots)$. In the case in which the λ_n are complex, the convergence-domains of (1.1) are not generally half-planes, and so the convergence-abscissas of (1.1) have no meaning.

The main theorems are:

THEOREM I. *In the case* $\phi_n = 0$ $(n=1, 2, \cdots)$, (1.1) *has three convergence-abscissas, i.e. a simple convergence-abscissa* σ_s, *a uniform convergence-abscissa* σ_u, *and an absolute convergence-abscissa* σ_a *such that* $\sigma_s = \sigma_u \leqq \sigma_a$.

REMARK. (1) In the convergence-problem of (1.1), the sequence of points $-\lambda_n$ $(n=1, 2, \cdots)$ is excluded from the s-plane by small circles with centres at $-\lambda_n$ $(n=1, 2, \cdots)$ and radii ϵ, ϵ being a small positive constant.

(2) The divergence of $\sum_{n=1}^{\infty} 1/r_n$ is not necessary for the validity of Theorem 1.

THEOREM II. *If* $\sum_{n=1}^{\infty} 1/r_n < +\infty$, *the necessary and sufficient condition for* (1.1) *to be simply* (*absolutely*) *convergent at* $s = s_0$ *distinct from* $-\lambda_n$ $(n=1, 2, \cdots)$ *is that* $\sum_{n=1}^{\infty} a_n$ ($\sum_{n=1}^{\infty} |a_n|$) *converges. If*

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furthermore $\phi_n = 0$ $(n = 1, 2, \cdots)$, *then three possibilities now present themselves*:

Case	$\sum_{n=1}^{\infty} \vert a_n \vert$	$\sum_{n=1}^{\infty} a_n$	$\sigma_s = \sigma_u$	σ_a
I	$< + \infty$	convergent	$= - \infty$	$= - \infty$
II	$= + \infty$	convergent	$= - \infty$	$= + \infty$
III	$= + \infty$	divergent	$= + \infty$	$= + \infty$

THEOREM III. *If* $\sum_{n=1}^{\infty} 1/r_n = +\infty$, $\phi_n = 0$ $(n = 1, 2, \cdots)$, *then the three convergence-abscissas of* (1.1) *are determined respectively by*

(a) $$\sigma_s = \sigma_u = \limsup_{n\to\infty} 1/l_n \cdot \log \left| \sum_{\nu=1}^{n} a_\nu \exp\left(\phi(l_\nu) - \phi(l_n)\right) \right|,$$

(b) $$\sigma_a = \limsup_{n\to\infty} 1/l_n \cdot \log \left\{ \sum_{\nu=1}^{n} \vert a_\nu \vert \exp\left(\phi(l_\nu) - \phi(l_n)\right) \right\},$$

where

(c) $$l_n = \sum_{i=1}^{n} l/r_i \qquad (0 < l_1 < l_2 < \cdots < l_n \to +\infty),$$

(d) $\phi(x)$ *is the positive and differentiable function defined for* $x > 0$ *such that*

(i) $\phi(x) \uparrow +\infty$, $\phi'(x) \to +\infty$ *as* $x \to +\infty$.

(ii) *for any given* $\epsilon > 0$, $\int^{+\infty} \exp(-\epsilon x) \vert \phi'(x) \vert dx < +\infty$.

COROLLARY I (EQUICONVERGENCE THEOREM) (T. Fort [4, p. 239]). *Under the same conditions as in Theorem* III, (1.1) *has the same abscissa of simple convergence and the same abscissa of absolute convergence as the Dirichlet series*

(1.3) $$G(s) = \sum_{n=1}^{\infty} a_n \exp(-l_n s).$$

COROLLARY II. *Under the same conditions as in Theorem* III, *we have*

(a) $$\sigma_s = \sigma_u = \limsup_{x\to\infty} 1/x \cdot \log \left| \sum_{[x] \le l_n < x} a_n \right|,$$

$$\sigma_a = \limsup_{x\to\infty} 1/x \cdot \log \left\{ \sum_{[x] \le l_n < x} \vert a_n \vert \right\},$$

where $[x]$ *denotes the greatest integer contained in* x.

(b) $$0 \leqq \sigma_a - \sigma_s \leqq \limsup_{n\to\infty} 1/l_n \cdot \log n.$$

2. **Proof of Theorem I.** We first prove some necessary lemmas, which are analogues of theorems concerning ordinary factorial series [**6**, pp. 171–174].

LEMMA I. *If* (1.1) *is simply convergent at* $s=s_0$, *then* (1.1) *is uniformly convergent in the angular domain* $D(s_0, \vartheta, \phi)$: $|\arg (s-s_0)| \leqq \vartheta < (\pi/2-\phi)$, *where* ϑ *is an arbitrary but fixed constant.*

As a special case of Lemma I, we have

LEMMA I′. *If* (1.1) *with real* λ_n $(n=1, 2, \cdots)$ *is simply convergent at* $s=s_0$, *then* (1.1) *is uniformly convergent in the angular domain* $D(s_0, \vartheta, 0)$: $|\arg (s-s_0)| \leqq \vartheta < \pi/2$, *where* ϑ *is an arbitrary but fixed constant.*

Under the assumptions that $\lim_{n\to\infty} \phi_n = 0$, and $\sum_{n=1}^{\infty} 1/r_n = +\infty$, T. Fort [**4**, p. 237, Theorem IV] has proved that (1.1) converges uniformly in the angular domain $D(s_0, \vartheta, 0)$, provided that it converges simply at $s=s_0$. Since we can put $\phi=\epsilon$ in Lemma I, ϵ being any small positive constant, this theorem is evidently contained in Lemma I.

Proof of Lemma I. We first establish the inequality

$$|s+\lambda_n| > |s_0+\lambda_n| + r \sin(\eta/2) \qquad \text{for } n \geqq n_1, \tag{2.1}$$

where

(i) $s \in D(s_0, \vartheta, \phi)$, $r=|s-s_0|$, $\vartheta = \pi/2-(\phi+\eta)$ $(\eta>0)$,

(ii) n_1 is a sufficiently large integer.

In fact, putting $\theta = \arg (s-s_0) - \arg (s_0+\lambda_n)$, where $s \in D(s_0, \vartheta, \phi)$, we have easily

$$\pi/2 + \eta/2 \leqq \theta < 3\pi/2 - \eta/2 \qquad \text{for } n \geqq n_1,$$

so that

$$\begin{aligned} |s+\lambda_n|^2 &= r^2 + |s_0+\lambda_n|^2 - 2r|s_0+\lambda_n| \cos\theta \\ &\geqq \{|s_0+\lambda_n| + r\sin(\eta/2)\}^2 \qquad \text{for } n \geqq n_1, \end{aligned}$$

which proves (2.1). Let us put

$$b_n = a_n[\lambda_1 \cdots \lambda_n][(s_0+\lambda_1)(s_0+\lambda_2)\cdots(s_0+\lambda_n)]^{-1},$$

$$c_n(s) = [(s_0+\lambda_1)(s_0+\lambda_2)\cdots(s_0+\lambda_n)][(s+\lambda_1)(s+\lambda_2)\cdots (s+\lambda_n)]^{-1}. \tag{2.2}$$

Equation (2.1) yields

$$(2.3)\quad \begin{aligned} &|(s_0+\lambda_n)/(s+\lambda_n)| < \rho_n[\rho_n + r\sin(\eta/2)]^{-1}, \\ &|(s-s_0)/(s+\lambda_{n+1})| < r[\rho_{n+1} + r\sin(\eta/2)]^{-1}, \end{aligned} \quad \text{for } n \geqq n_1,$$

where $s \in D(s_0, \vartheta, \phi)$, $r=|s-s_0|$, and $\rho_n=|s_0+\lambda_n|$. Hence

$$(2.4)\quad \begin{aligned} |c_n(s) - c_{n+1}(s)| &= |c_n(s)(s-s_0)(s+\lambda_{n+1})^{-1}| \\ &< |K(s)| \cdot d_n \cdot r[\rho_{n+1} + r\sin(\eta/2)]^{-1}, \end{aligned}$$

where

$$K(s) = [(s_0+\lambda_1)(s_0+\lambda_2)\cdots(s_0+\lambda_{n_1-1})][(s+\lambda_1)(s+\lambda_2)\cdots(s+\lambda_{n_1-1})]^{-1},$$

and

$$d_n = \prod_{i=n_1}^{n} \rho_n[\rho_n + r\sin(\eta/2)]^{-1}.$$

In D_0, which we get by excluding from $D(s_0, \vartheta, \phi)$ the sequence of circles with centres at $-\lambda_n$ $(n=1, 2, \cdots)$ and radii ϵ, ϵ being a small positive constant, we have evidently

$$(2.5)\quad |K(s)| < K,$$

where K is a suitable constant. Since

$$d_n \cdot r \cdot [\rho_{n+1} + r\sin(\eta/2)]^{-1} = \operatorname{cosec}(\eta/2)(d_n - d_{n+1}),$$

taking account of (2.4) and (2.5), we have for any large N

$$(2.6)\quad \begin{aligned} \sum_{n=n_1}^{N} |c_n(s) - c_{n+1}(s)| &< K \operatorname{cosec}(\eta/2) \sum_{n=n_1}^{N} (d_n - d_{n+1}) \\ &< K \operatorname{cosec}(\eta/2) d_{n_1} \end{aligned}$$

uniformly in D_0.

Since $\sum_{n=1}^{\infty} b_n$ is convergent by the hypothesis, on account of (2.6) and du Bois-Reymond's Theorem [7, p. 315], $F(s) = \sum_{n=1}^{\infty} b_n c_n(s)$ is uniformly convergent in D_0. q.e.d.

LEMMA II. *If* (1.1) *is absolutely convergent at* $s=s_0$, *then* $\sum_{n=1}^{\infty} |a_n| |(\lambda_1\lambda_2\cdots\lambda_n)[(s+\lambda_1)(s+\lambda_2)\cdots(s+\lambda_n)]^{-1}|$ *is uniformly convergent in the angular domain* $D(s_0, \vartheta, \phi)$, *where* $D(s_0, \vartheta, \phi)$ *has the same meaning as in Lemma* I.

As a corollary, we get

LEMMA II′. *If* (1.1) *with real* λ_n $(n=1, 2, \cdots)$ *is absolutely convergent at* $s=s_0$, *then* (1.1) *is absolutely and uniformly convergent in the* *angular domain* $D(s_0, \vartheta, 0)$.

Proof of Lemma II. Using the same notation as in Lemma I, (2.1) and (2.3) are also valid. Since

$$\left| |c_n(s)| - |c_{n+1}(s)| \right| = |c_n(s)| \cdot \left| 1 - |(s_0 + \lambda_{n+1})(s + \lambda_{n+1})^{-1}| \right|$$
$$\leqq |c_n(s)| \cdot |(s - s_0)(s + \lambda_{n+1})^{-1}|,$$

on account of (2.4) and (2.5), we obtain for any large N

$$(2.7) \qquad \sum_{n=n_1}^{N} \left| |c_n(s)| - |c_{n+1}(s)| \right| < K \operatorname{cosec}(\eta/2) \cdot d_{n_1}$$

uniformly in D_0. Since $\sum_{n=1}^{\infty} |b_n|$ is convergent by the hypothesis, it results by virtue of (2.7) and du Bois-Reymond's theorem that $\sum_{n=1}^{\infty} |b_n \cdot c_n(s)|$ is uniformly convergent in D_0. q.e.d.

LEMMA III. *If* (1.1) *is simply convergent at* $s = s_0$, *and furthermore there exists a point* s_1 *contained in the angular domain* $D(s_0, \pi/2 - \phi)$: $|\arg (s - s_0)| \leqq \pi/2 - \phi$, *such that for a sufficiently large integer* n_1, *we have*

$$|\arg (s_1 + \lambda_n)| \leqq \phi \qquad \text{for } n \geqq n_1,$$

then (1.1) *is uniformly convergent in the angular domain* $D(s_2, \pi/2 - \phi)$, *where* $s_2 = s_1 + \epsilon \sec \phi$, ϵ *being any small positive constant.*

As an immediate consequence of Lemma III, we get

LEMMA III′. *If* (1.1) *with real* λ_n $(n = 1, 2, \cdots)$ *is simply convergent at* $s = s_0$, *then* (1.1) *is uniformly convergent in the half-plane* D: $\Re(s) \geqq \Re(s_0) + \epsilon$, ϵ *being any small positive constant.*

In fact, we can put $\phi = 0$, $s_1 = \Re(s_0)$, and $s_2 = \Re(s_0) + \epsilon$ in Lemma III.

Proof of Lemma III. We first prove

$$(2.8) \qquad |s + \lambda_n| \geqq |s_3 + \lambda_n| + \epsilon/2 \qquad \text{for } n \geqq n_1,$$

where $s \in D(s_2, \pi/2 - \phi)$, and $s_3 = s_1 + \epsilon/2 \cdot \sec \phi$. In fact, putting $\alpha_n = \arg (s_3 + \lambda_n)$, we have evidently

$$(2.9) \qquad |\alpha_n| \leqq \phi \qquad \text{for } n \geqq n_1.$$

Projecting the vector $(s + \lambda_n)$ perpendicularly on the vector $(s_3 + \lambda_n)$, we get easily

$$|s + \lambda_n| \geqq |s_3 + \lambda_n| + \epsilon/2 \cdot \sec \phi \cdot \cos \alpha_n,$$

so that, by (2.9),

$$|s + \lambda_n| \geqq |s_3 + \lambda_n| + \epsilon/2,$$

which proves (2.8).

Let us put

$$b_n = a_n[\lambda_1 \cdots \lambda_n][(s_3 + \lambda_1)(s_3 + \lambda_2) \cdots (s_3 + \lambda_n)]^{-1},$$

$$\text{(2.10)}\quad c_n(s) = [(s_3 + \lambda_1)(s_3 + \lambda_2) \cdots (s_3 + \lambda_n)] \cdot [(s + \lambda_1)(s + \lambda_2) \cdots (s + \lambda_n)]^{-1}.$$

By (2.8) and arguments similar to those employed in the proof of Lemma I, we have

$$\text{(2.11)}\quad \begin{aligned} | c_n(s) - c_{n+1}(s) | &= | c_n(s) | \cdot | (s - s_3)(s + \lambda_{n+1})^{-1} | \\ &< | K(s) | \cdot d_n \cdot (\rho_n + \epsilon/2)^{-1}, \end{aligned}$$

where

$$K(s) = (s - s_3)[(s_3 + \lambda_1) \cdots (s_3 + \lambda_{n_1-1})] \cdot [(s + \lambda_1) \cdots (s + \lambda_{n_1-1})]^{-1},$$

$$\rho_n = | s_3 + \lambda_n |, \qquad d_n = \prod_{i=n_1}^{n} \rho_i(\rho_i + \epsilon/2)^{-1}.$$

Since $d_n(\rho_n+\epsilon/2)^{-1} = 2/\epsilon \cdot (d_n - d_{n+1})$, and $K(s) = O(1)$ in the domain D_0, as is easily seen by excluding from $D(s_2, \pi/2-\phi)$ the sequence of small circles with centres at $-\lambda_n$ $(n=1, 2, \cdots)$ and radii $\epsilon'>0$, by virtue of (2.11) we have

$$| c_n(s) - c_{n+1}(s) | < 2K/\epsilon \cdot (d_n - d_{n+1}) \qquad \text{for } n \geqq n_1,$$

uniformly in D_0, where K is a suitable constant. Hence

$$\text{(2.12)}\quad \sum_{n=n_1}^{N} | c_n(s) - c_{n+1}(s) | < 2K/\epsilon \cdot (d_{n_1} - d_{N+1}) < 2K/\epsilon \cdot d_{n_1}$$

uniformly in D_0 for any given N.

Since (1.1) is simply convergent at $s=s_0$ by virtue of Lemma I, it follows from (2.12) and du Bois-Reymond's theorem that $F(s) = \sum_{n=1}^{\infty} b_n c_n(s)$ is uniformly convergent in D_0. q.e.d.

Now we are in a position to prove Theorem I.

Proof of Theorem I. If (1.1) is simply (absolutely) convergent at $s=s_0$, then (1.1) is also simply (absolutely) convergent at $s=s_1$ with $\Re(s_0) < \Re(s_1)$ by virtue of Lemma I′ (Lemma II″). Hence there exists a simple (absolute) convergence-abscissa $\sigma_s(\sigma_a)$ of (1.1), and we have evidently $\sigma_s \leqq \sigma_a$.

For any given $\epsilon>0$, (1.1) is simply convergent at $s=\sigma_s+\epsilon/2$, so that by Lemma III′, (1.1) is uniformly convergent for $\Re(s) \geqq \sigma_s+\epsilon$. But since (1.1) is not simply convergent on $s=\sigma_s-\epsilon$, (1.1) is not uniformly convergent for $\Re(s) \geqq \sigma_s - \epsilon$. Hence σ_u coincides with σ_s. Thus we have $\sigma_s = \sigma_u \leqq \sigma_a$. q.e.d.

3. **Proof of Theorem II.** Since $\sum_{n=1}^{\infty} 1/r_n < +\infty$, the infinite product $g(s) = \prod_{n=1}^{\infty} (1+s/\lambda_n)$ converges, so that we have

$$(3.1) \qquad 0 < |g(s)| < +\infty \qquad \text{for } s \neq -\lambda_n \ (n = 1, 2, \cdots).$$

Let us put

$$c_n(s) = [\lambda_1 \cdots \lambda_n][(s + \lambda_1)(s + \lambda_2) \cdots (s + \lambda_n)]^{-1} = [g_n(s)]^{-1},$$

where $g_n(s) = \prod_{i=1}^{n} (1+s/\lambda_i)$. Since

$$c_n(s) - c_{n+1}(s) = [g_n(s) \cdot \lambda_{n+1}]^{-1} \cdot s(1 + s/\lambda_{n+1})^{-1},$$

by (3.1) we get

$$|c_n(s) - c_{n+1}(s)| < K_1 |g(s)|^{-1} \cdot 1/r_{n+1} \qquad \text{for } n \geqq n_1,$$

where (i) K_1 is a suitable constant, (ii) n_1 is a sufficiently large integer. Hence

$$(3.2) \qquad \sum_{n=n_1}^{\infty} |c_n(s) - c_{n+1}(s)| < K_1 |g(s)|^{-1} \cdot \sum_{n=n_1}^{\infty} 1/r_{n+1} < +\infty.$$

If $\sum_{n=1}^{\infty} a_n$ converges, then by (3.2) and du Bois-Reymond's theorem, $F(s) = \sum_{n=1}^{\infty} a_n c_n(s)$ also converges for s different from $-\lambda_n$ $(n=1, 2, \cdots)$.

Next suppose that $F(s_0) = \sum_{n=1}^{\infty} b_n(s_0)$ converges for $s = s_0 \neq -\lambda_n$ $(n=1, 2, \cdots)$, where

$$b_n(s_0) = a_n [\lambda_1 \cdots \lambda_n][(s_0 + \lambda_1)(s_0 + \lambda_2) \cdots (s_0 + \lambda_n)]^{-1}.$$

Since $g_{n+1}(s_0) - g_n(s_0) = g_n(s_0) \cdot s_0/\lambda_{n+1}$, by (3.1) we get

$$|g_{n+1}(s_0) - g_n(s_0)| < |g(s_0)| \cdot K_2/r_{n+1} \qquad \text{for } n \geqq n_2,$$

where (i) K_2 is a suitable constant, (ii) n_2 is a sufficiently large integer, so that

$$(3.3) \qquad \sum_{n=n_2}^{\infty} |g_{n+1}(s_0) - g_n(s_0)| < |g(s_0)| \cdot K_2 \cdot \sum_{n=n_2}^{\infty} 1/r_{n+1} < +\infty.$$

Since $\sum_{n=1}^{\infty} b_n(s_0)$ converges, by (3.3) and du Bois-Reymond's theorem, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n(s_0) g_n(s_0)$ is also convergent.

By entirely similar arguments, we can prove that the necessary-sufficient condition for (1.1) to converge absolutely at $s = s_0$ different from $-\lambda_n$ $(n=1, 2, \cdots)$ is that $\sum_{n=1}^{\infty} |a_n| < +\infty$.

If $\sum_{n=1}^{\infty} 1/r_n < +\infty$ and $\phi_n = 0$ $(n=1, 2, \cdots)$, then the second part of Theorem II immediately follows from Theorem I and what is proved above.

4. **Proof of Theorem III.** Let us put

$$k = \limsup_{n\to\infty} 1/l_n \cdot \log \left| \sum_{\nu=1}^{n} a_\nu \exp\left(\phi(l_\nu) - \phi(l_n)\right) \right|. \tag{4.1}$$

We shall first establish the inequality

$$k \leqq \sigma_s. \tag{4.2}$$

Since (1.1) is simply convergent for $s = \sigma > \sigma_s$, there exists a constant K such that

$$|S_n| < K \qquad (n = 1, 2, \cdots), \tag{4.3}$$

where

$$S_n = \sum_{i=1}^{n} a_i [\lambda_1 \cdots \lambda_i][(\sigma + \lambda_1)(\sigma + \lambda_2) \cdots (\sigma + \lambda_i)]^{-1}.$$

Putting $S_0 = 0$ and applying Abel's transformation, we have

$$\sum_{i=1}^{n} a_i \exp\left(\phi(l_i)\right) = \sum_{i=1}^{n-1} S_i(f(i) - f(i+1)) + S_n f(n), \tag{4.4}$$

where $f(i) = \exp\left(\phi(l_i)\right) \cdot \prod_{k=1}^{i} (1 + \sigma/\lambda_k)$. On the other hand,

$$f(i) = Q(\sigma) \cdot \exp\left\{\phi(l_i) + l_i(\sigma + \epsilon_i(\sigma))\right\} \qquad \text{for } i > n_1, \tag{4.5}$$

where

(i) $\quad Q(\sigma) = \prod_{n=1}^{n_1} (1 + \sigma/\lambda_n) \exp(-\sigma/\lambda_n),$

(ii) $\quad \lim_{i\to\infty} \epsilon_i(\sigma) = 0,$

(iii) $\quad n_1$ is a sufficiently large integer.

In fact, since

$$(1 + x) = \exp\left(x + x^2 \cdot \rho(x)\right), \quad |\rho(x)| \leqq 1 \qquad \text{for } |x| \leqq 1/2,$$

we can easily obtain the relation

$$\begin{aligned} f(i) &= \prod_{n=1}^{n_1} (1 + \sigma/\lambda_n) \exp(-\sigma/\lambda_n) \\ &\quad \times \exp\left\{\phi(l_i) + \sigma l_i + \sigma^2 \cdot \vartheta(\sigma)\left(\sum_{n=1}^{i} 1/\lambda_n^2\right)\right\}, \end{aligned} \tag{4.6}$$

where (i) $|\sigma/\lambda_n| \leqq 1/2$ for $n > n_1$, (ii) $|\vartheta(\sigma)| \leqq 1$. Since $\lim_{i\to\infty} 1/l_i \cdot \sum_{n=1}^{i} 1/\lambda_n^2 = 0$, (4.6) gives (4.5).

Taking account of the hypothesis (d) part (i), we can easily prove that

$$g(i) \uparrow \infty \qquad \text{for } i > n_2,$$

where

(i) $g(i) = \exp(\phi(l_i) + l_i(\sigma + \epsilon_i(\sigma)))$,

(ii) n_2 is a sufficiently large integer.

Therefore, putting $N = \operatorname{Max}(n_1, n_2)$, by (4.4) and (4.3) we have

$$\left| \sum_{i=1}^{n} a_i \exp(\phi(l_i)) \right| \leqq K \cdot \left| \sum_{i=1}^{N} f(i) - f(i+1) \right| + K \, | Q(\sigma) | \cdot \left\{ \sum_{i=N+1}^{n-1} g(i+1) - g(i) + g(n) \right\},$$

so that for sufficiently large n,

$$\left| \sum_{i=1}^{n} a_i \exp(\phi(l_i)) \right| < 3K \cdot | Q(\sigma) | \cdot g(n).$$

Hence $k \leqq \sigma + \lim_{n\to\infty} \epsilon_n(\sigma) = \sigma$. Letting $\sigma \to \sigma_s$, we have $k \leqq \sigma_s$, which proves (4.2).

Next we shall prove

$$\sigma_s \leqq k. \tag{4.7}$$

By the definition of k, for any given $\delta > 0$, there exists a constant N such that

$$| T_n | < U_n = \exp\{\phi(l_n) + l_n(k + \delta/2)\} \qquad \text{for } n \geqq N, \tag{4.8}$$

where $T_n = \sum_{k=1}^{n} a_i \exp(\phi(l_i))$. Taking account of $a_n = (T_n - T_{n-1}) \exp(-\phi(l_n))$, by Abel's transformation we get

$$\begin{aligned} &\sum_{i=N+1}^{M} a_i[\lambda_1 \cdots \lambda_i][(k+\delta+\lambda_1)(k+\delta+\lambda_2) \cdots (k+\delta+\lambda_i)]^{-1} \\ &= \sum_{i=N+1}^{M-1} T_i(h(i) - h(i+1)) - T_N h(N+1) + T_M h(M), \end{aligned} \tag{4.9}$$

where $h(i) = \exp(-\phi(l_i)) \cdot [\prod_{k=1}^{i} (1 + (k+\delta)/\lambda_k)]^{-1}$. By arguments similar to those employed before we may write

$$h(i) = K \cdot g(i) \qquad \text{for } i > n_1, \tag{4.10}$$

where

(i) $$K = \left[\prod_{n=1}^{n_1} (1 + (k + \delta)/\lambda_n) \cdot \exp(-(k + \delta)/\lambda_n) \right]^{-1},$$

(ii) $$g(i) = \exp\{-(\phi(l_i) + l_i(k + \delta + \epsilon_i))\},$$

(iii) $$\lim_{i\to\infty} \epsilon_i = 0,$$

(iv) n_1 is a sufficiently large integer.

Accordingly, by (4.8), (4.9), and (4.10) we obtain

$$(4.11)\quad \begin{aligned} &\left| \sum_{i=N+1}^{M} a_i[\lambda_1 \cdots \lambda_i][(k+\delta+\lambda_1)(k+\delta+\lambda_2) \cdots (k+\delta+\lambda_i)]^{-1} \right| \\ &\leqq |K| \left\{ \sum_{i=N+1}^{M-1} U_i |g(i)-g(i+1)| + U_N g(N+1) + U_M g(M) \right\}. \end{aligned}$$

On the other hand, for sufficiently large i, we get easily

$$\begin{aligned} |g(i) - g(i + 1)| &= O\left(\left| \int_{l_i}^{l_{i+1}} \frac{d}{dx} \exp(-(\phi(x) + x(k + \delta)))dx \right| \right) \\ &= O\left(1/U_i \cdot \int_{l_i}^{l_{i+1}} \exp(-\delta/2 \cdot x) |\phi'(x)| dx \right). \end{aligned}$$

Hence, by (4.9), (4.10) and the hypothesis (d) part (ii), we get for sufficiently large N

$$\begin{aligned} &\left| \sum_{i=N+1}^{M} a_i[\lambda_1 \cdots \lambda_i][(k + \delta + \lambda_1)(k + \delta + \lambda_2) \cdots (k + \delta + \lambda_i)]^{-1} \right| \\ &= O\left(\int_{l_{M+1}}^{l_M} \exp(-\delta/2 \cdot x) |\phi'(x)| dx \right) + O(\exp(-l_{N+1}(\delta/2 + \epsilon_{N+1}))) \\ &\quad + O(\exp(-l_M(\delta/2 + \epsilon_M))) = o(1), \end{aligned}$$

so that (1.1) is simply convergent at $s = k + \delta$. Therefore

$$\sigma_s < k + \delta$$

for any given $\delta > 0$, which proves (4.6).

Thus, by (4.3), (4.7), and Theorem 1, we have

$$k = \sigma_s = \sigma_u,$$

which proves (a) of Theorem III. By the slight modification of the above arguments, we can also prove (b) of Theorem III.

5. **Proof of corollaries.** By M. Fujiwara's theorem [8], the simple convergence-abscissa $\sigma_s(G)$ and the absolute convergence-abscissa

$\sigma_a(G)$ of $G(s)$ are given respectively by

$$\sigma_s(G) = \limsup_{n\to\infty} 1/l_n\cdot\log\left|\sum_{\nu=1}^{n} a_\nu \exp(l_\nu^2 - l_n^2)\right|, \tag{5.1}$$
$$\sigma_a(G) = \limsup_{n\to\infty} 1/l_n\cdot\log\left\{\sum_{\nu=1}^{n} |a_\nu| \exp(l_\nu^2 - l_n^2)\right\}.$$

Since $\phi(x)=x^2$ evidently satisfies the conditions of Theorem III, taking account of Theorem III and (5.1) we get

$$\sigma_s = \sigma_s(G), \qquad \sigma_a = \sigma_a(G),$$

which proves Corollary I.

By T. Kojima's theorem [9], we may write

$$\sigma_s(G) = \limsup_{x\to\infty} 1/x\cdot\log\left|\sum_{[x]\leqq l_n<x} a_n\right|,$$
$$\sigma_a(G) = \limsup_{x\to\infty} 1/x\cdot\log\left\{\sum_{[x]\leqq l_n<x} |a_n|\right\},$$

so that the first part of Corollary II follows immediately from Corollary I. On the other hand, by a well known theorem [10, p. 49], we have

$$0 \leqq \sigma_a(G) - \sigma_s(G) \leqq \limsup_{n\to\infty} 1/1_n\cdot\log n,$$

which proves the second part of corollary II.

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