ON SPECTRAL PERMANENCE FOR CERTAIN BANACH ALGEBRAS

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Introduction. Let $\mathfrak{A}$ be a commutative Banach algebra which is *algebraically* embedded in a second such Banach algebra $\mathfrak{B}$. Let $\phi: x \rightarrow x(\phi)$ denote a homomorphism of $\mathfrak{A}$ into the complex numbers. The main problem considered here is that of extending ϕ to all of $\mathfrak{B}$. This extension is, of course, not always possible. On the other hand, it has been proved by Šilov [**3**; **12**] that certain ϕ can be extended to $\mathfrak{B}$ in case the embedding of $\mathfrak{A}$ in $\mathfrak{B}$ is an isometry. It is proved below (Theorem 1) that the extension can always be made if $\mathfrak{A}$ is semi-simple and regular in a sense defined by Šilov [**13**]. This implies that elements of $\mathfrak{A}$ have the same spectra in $\mathfrak{A}$ as in $\mathfrak{B}$, and also has as a corollary a result due to Kaplansky [**5**, Theorem 6.2] concerning the minimal character of the norm in an algebra $C(\Omega)$. In §2 some results for the noncommutative case are obtained when $\mathfrak{A}$ is a B^*-algebra [**9**]. For example, if $\mathfrak{A}$ is a B^*-algebra which is algebraically embedded in an arbitrary Banach algebra $\mathfrak{B}$ and x is an element of $\mathfrak{A}$, then the spectrum of x in $\mathfrak{A}$ is equal to the join of the spectrum of x and the conjugate of the spectrum of x^* in $\mathfrak{B}$. Also, the spectral radii of x in $\mathfrak{A}$ and in $\mathfrak{B}$ are always equal. If in addition the involution in $\mathfrak{A}$ can be extended to $\mathfrak{B}$, then the spectra of x in $\mathfrak{A}$ and in $\mathfrak{B}$ are equal.

1. **Regular Banach algebras.** Let $\mathfrak{A}$ be a real or complex commutative Banach algebra, not necessarily with an identity element. Let $\Phi_{\mathfrak{A}}$ be the set of all (algebra) homomorphisms of $\mathfrak{A}$ *into* the complex numbers. The zero homomorphism is included here and will be denoted by ϕ_∞. The image of $x \in \mathfrak{A}$ under $\phi \in \Phi_{\mathfrak{A}}$ is denoted by $x(\phi)$. If $\mathfrak{A}$ is complex, then every element of $\Phi_{\mathfrak{A}}$ different from ϕ_∞ maps $\mathfrak{A}$ *onto* the complex numbers. If $\mathfrak{A}$ is real, then an element of $\Phi_{\mathfrak{A}}$ may map $\mathfrak{A}$ onto either the reals or complexes. The kernel of any ϕ different from ϕ_∞ is a maximal regular[1] ideal in $\mathfrak{A}$. Conversely, every maximal regular ideal in $\mathfrak{A}$ is the kernel of an element of $\Phi_{\mathfrak{A}}$ different from ϕ_∞. However, if $\mathfrak{A}$ is real, this correspondence between elements of $\Phi_{\mathfrak{A}} - \phi_\infty$ and maximal regular ideals need not be one-to-one.[2]

In the usual way, $\Phi_{\mathfrak{A}}$ is made into a topological space *via* the funda-

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[1] An ideal I is *regular* provided there exists an element e which is an identity modulo I; i.e., $ex - x$ belongs to I for every x [**11**].

[2] Consider, for example, a ϕ which maps $\mathfrak{A}$ onto the complex numbers and define ϕ' by $x(\phi') = \overline{x(\phi)}$. Then $\phi' \in \Phi_{\mathfrak{A}}$, $\phi' \neq \phi$, but ϕ and ϕ' have the same kernel.

mental system of neighborhoods

$$N_\phi(a_1, \cdots, a_n; \epsilon) = \{\phi' \mid |a_i(\phi') - a_i(\phi)| < \epsilon, i = 1, \cdots, n, \phi' \in \Phi_{\mathfrak{A}}\},$$

where $\epsilon > 0$ and $a_1, \cdots, a_n$ is any finite set of elements in $\mathfrak{A}$. Under this topology, $\Phi_{\mathfrak{A}}$ becomes a compact Hausdorff space called the *structure space* of $\mathfrak{A}$. If $\mathfrak{A}$ has an identity element, then ϕ_∞ is an isolated point of $\Phi_{\mathfrak{A}}$. If $\mathfrak{A}$ does not have an identity and $\mathfrak{A}'$ is the algebra obtained in the usual way by adjoining to $\mathfrak{A}$ an identity element, then it is easy to see that $\Phi_{\mathfrak{A}'}$ can be obtained from $\Phi_{\mathfrak{A}}$ by adding a single isolated point, the zero homomorphism of $\mathfrak{A}'$. If $\mathfrak{A}$ is complex, then $\Phi_{\mathfrak{A}} - \phi_\infty$ can be identified with the space of maximal regular ideals of $\mathfrak{A}$ as usually defined. If $\mathfrak{A}$ is real and $\mathfrak{A}^c$ is the complexification [5] of $\mathfrak{A}$, then $\Phi_{\mathfrak{A}}$ and $\Phi_{\mathfrak{A}^c}$ are homeomorphic under the natural mapping from $\Phi_{\mathfrak{A}^c}$ to $\Phi_{\mathfrak{A}}$ obtained by restricting elements of $\Phi_{\mathfrak{A}^c}$ to $\mathfrak{A}$.

For each x in $\mathfrak{A}$, $x(\phi)$ is a continuous function of ϕ which vanishes at the "point at infinity" ϕ_∞. The set of nonzero points in the range of $x(\phi)$ coincides with the set of nonzero points in the spectrum[3] $\sigma_{\mathfrak{A}}(x)$. For the sake of simplicity, we shall include zero in the spectrum in all cases so that the range of $x(\phi)$ is always equal to $\sigma_{\mathfrak{A}}(x)$. In any case, the spectral radius[4] $r_{\mathfrak{A}}(x)$ is given by $\lim \|x^n\|^{1/n}$ as $n \to \infty$ [6, p. 818]. An element such that $x(\phi) = 0$ for every ϕ is necessarily quasi-regular. Semi-simplicity of $\mathfrak{A}$ is equivalent to the condition that $x(\phi) = 0$ for every ϕ imply $x = 0$.

The algebra $\mathfrak{A}$ is called *regular*[5] provided, for every closed set $F \subset \Phi_{\mathfrak{A}}$ and $\phi_0 \in \Phi_{\mathfrak{A}} - F$, there exists $u \in \mathfrak{A}$ such that $u(\phi)$ is constant on F with value different from $u(\phi_0)$. If $\mathfrak{A}$ is complex, then an equivalent definition is that, for every closed set $F \subset \Phi_{\mathfrak{A}}$ and $\phi_0 \in \Phi_{\mathfrak{A}} - F$ with $\phi_0 \neq \phi_\infty$, there exist $u \in \mathfrak{A}$ such that $u(\phi_0) = 1$ and $u(\phi) = 0$ for[6] $\phi \in F$. The latter definition is the one given by Šilov [12] for complex $\mathfrak{A}$

[3] In defining the spectrum, we prefer the operation $x \circ y = x + y - xy$ used by Segal [11] and Hille [4, p. 455] rather than $x + y + xy$ used by some algebraists. An element x is called *quasi-regular* if there exists y such that $x \circ y = y \circ x = 0$. Recall that, for complex algebras, a nonzero complex number λ is in the spectrum of x provided $\lambda^{-1}x$ is not quasi-regular [4, p. 458]. We refer to [6] or [10] for the definition in the real case.

[4] The spectral radius is $r_{\mathfrak{A}}(x) = \max |\lambda|, \lambda \in \sigma_{\mathfrak{A}}(x)$.

[5] This concept of regularity should not be confused with that introduced by von Neumann [8].

[6] It is obvious that the first definition always implies the second. In order to show that the second implies the first, it is sufficient to prove, for every closed set $F \subset \Phi_{\mathfrak{A}}$ with $\phi_\infty \notin F$, the existence of $u \in \mathfrak{A}$ such that $u(\phi) = 1$ for $\phi \in F$. This is, of course, trivial if $\mathfrak{A}$ has an identity element. In case $\mathfrak{A}$ does not have an identity, the proof that such a u exists, which seems to depend essentially on $\mathfrak{A}$ being complex, will be found in [7, pp. 105–110].

with an identity element. In any case, if $\mathfrak{A}$ is regular, then it is *normal* in the following sense: If F_0 and F_1 are disjoint closed sets in $\Phi_{\mathfrak{A}}$ with $\phi_\infty \notin F_1$, then there exists $u \in \mathfrak{A}$ such that $u(\phi)=0$ for $\phi \in F_0$ and $u(\phi)=1$ for $\phi \in F_1$. The proof of this fact follows essentially the same lines as the proof given by Šilov [13, p. 37] in the case considered by him.[7]

We turn now to the extension theorem mentioned in the introduction.

THEOREM 1. *Let $\mathfrak{A}$ be a semi-simple regular Banach algebra which is algebraically embedded in a second Banach algebra $\mathfrak{B}$. Then every ϕ in $\Phi_{\mathfrak{A}}$ can be extended to an element of $\Phi_{\mathfrak{B}}$.*

PROOF. Let Ψ denote the subset of $\Phi_{\mathfrak{A}}$ obtained by restricting elements of $\Phi_{\mathfrak{B}}$ to $\mathfrak{A}$. We have to prove $\Psi = \Phi_{\mathfrak{A}}$. Suppose that there exists an element $\phi_0 \in \Phi_{\mathfrak{A}} - \Psi$. Then $\phi_0 \neq \phi_\infty$ and, for every $\psi \in \Phi_{\mathfrak{B}}$, there exists $a_\psi \in \mathfrak{A}$ such that $a_\psi(\phi_0) \neq a_\psi(\psi)$. Set $\epsilon_\psi = |a_\psi(\psi) - a_\psi(\phi_0)|/2$ and define the set

$$G_\psi = \{\psi' \mid |a_\psi(\psi') - a_\psi(\phi_0)| > \epsilon_\psi, \psi' \in \Phi_{\mathfrak{B}}\}.$$

Then G_ψ is open in $\Phi_{\mathfrak{B}}$ and contains ψ. Since $\Phi_{\mathfrak{B}}$ is compact, there exists a finite number of sets $G_{\psi_1}, \cdots, G_{\psi_n}$ which cover $\Phi_{\mathfrak{B}}$. Set $a_i = a_{\psi_i}$, $\epsilon = \min_i \epsilon_{\psi_i}$, and consider the neighborhood

$$N_0 = N_{\phi_0}(a_1, \cdots, a_n; \epsilon)$$

in $\Phi_{\mathfrak{A}}$. Then N_0 is disjoint from Ψ. Now choose in $\Phi_{\mathfrak{A}}$ an open set G which contains ϕ_0 and whose closure is contained in N_0. By the normality of $\mathfrak{A}$, there exist elements $u, v \in \mathfrak{A}$ such that $u(\phi_0)=1$, $u(\phi)=0$ for $\phi \notin G$, $v(\phi)=1$ for $\phi \in G$, and $v(\phi)=0$ for $\phi \notin N_0$. Then $(uv)(\phi) = u(\phi)$, for all $\phi \in \Phi_{\mathfrak{A}}$, and therefore $uv=u$ by the semi-simplicity of $\mathfrak{A}$. Since $v(\psi)=0$ for every $\psi \in \Phi_{\mathfrak{B}}$, it follows that v is quasi-regular in $\mathfrak{B}$. In other words, there exists $w \in \mathfrak{B}$ such that $v \circ w = 0$. Observe that $uv=u$ implies $u \circ v = v$. Therefore $0 = v \circ w = (u \circ v) \circ w = u \circ (v \circ w) = u$. Since $u \neq 0$, this is a contradiction and completes the proof.

COROLLARY 1. *$\sigma_{\mathfrak{A}}(x) = \sigma_{\mathfrak{B}}(x)$ and hence $r_{\mathfrak{A}}(x) = r_{\mathfrak{B}}(x)$ for every $x \in \mathfrak{A}$.*

COROLLARY 2. *Let $\|x\|_1$ be any norm in $\mathfrak{A}$ under which $\mathfrak{A}$ is a (not necessarily complete) normed algebra. Then $r_{\mathfrak{A}}(x) \leq \|x\|_1$ for all x.*

If $\mathfrak{A}$ is taken to be the Banach algebra $C(\Omega)$ of all (real or complex) continuous functions vanishing at infinity on a locally compact

[7] A proof for the complex case without assumption of an identity is given by Mackey [7, pp. 105–110].

Hausdorff space Ω, then $\mathfrak{A}$ is semi-simple and regular, and Corollary 2 reduces to the Kaplansky result mentioned in the introduction. The above proof of Theorem 1 was, in fact, suggested by the Kaplansky proof.

2. **B^*-algebras.** A complex Banach algebra $\mathfrak{A}$, not necessarily commutative or with an identity, is called a B^*-algebra [9] provided it has an involution $x \rightarrow x^*$ (a conjugate-linear anti-automorphism of period two) with the property $\|xx^*\| = \|x\|^2$ for every x. An algebra of type $C(\Omega)$ is a (commutative) B^*-algebra with involution $x \rightarrow x^*$ defined by

$$x^*(\omega) = \overline{x(\omega)}, \quad \omega \in \Omega.$$

The important fact here is the converse [1; 2]. More precisely, every commutative B^*-algebra $\mathfrak{C}$ is isometrically $*$-isomorphic to $C(\Phi_{\mathfrak{C}} - \phi_\infty)$. In particular, commutative B^*-algebras are regular. Another consequence of this fact is that $\|xx^*\| = r(xx^*)$ for every x in $\mathfrak{A}$. It follows directly from the algebraic properties of an involution that an element x is quasi-regular if, and only if, each of the elements x^*, $x \circ x^*$, and $x^* \circ x$ is quasi-regular. Moreover, for every x, the spectrum of x^* is equal to the complex conjugate

$$\overline{\sigma_{\mathfrak{A}}(x)}$$

of the spectrum of x.

In the remainder of our discussion, $\mathfrak{A}$ will be a B^*-algebra which is algebraically embedded in a general Banach algebra $\mathfrak{B}$. The norm in $\mathfrak{A}$ will be denoted by $\|x\|$ and the norm in $\mathfrak{B}$ by $\|x\|_1$. The theorem which follows is an extension of a previous result of the author's [9, Theorem 1.6].

THEOREM 2. *In order for an element x of $\mathfrak{A}$ to be quasi-regular in $\mathfrak{A}$ it is necessary and sufficient that both x and x^* be quasi-regular in $\mathfrak{B}$.*

PROOF. The necessity is obvious. Hence assume x and x^* to be quasi-regular in $\mathfrak{B}$. Then both $x \circ x^*$ and $x^* \circ x$ are quasi-regular in $\mathfrak{B}$. Let $\mathfrak{C}$ be a maximal commutative B^*-subalgebra of $\mathfrak{A}$ which contains $x \circ x^*$, and let $\mathfrak{C}_1$ be a maximal commutative subalgebra of $\mathfrak{B}$ which contains $\mathfrak{C}$. Then both $\sigma_{\mathfrak{A}}(x \circ x^*) = \sigma_{\mathfrak{C}}(x \circ x^*)$ and $\sigma_{\mathfrak{C}_1}(x \circ x^*) = \sigma_{\mathfrak{B}}(x \circ x^*)$. Moreover, since $\mathfrak{C}$ is regular and is a subalgebra of $\mathfrak{C}_1$, it follows by Corollary 1 of Theorem 1 that $\sigma_{\mathfrak{C}}(x \circ x^*) = \sigma_{\mathfrak{C}_1}(x \circ x^*)$. Therefore $\sigma_{\mathfrak{A}}(x \circ x^*) = \sigma_{\mathfrak{B}}(x \circ x^*)$. This implies that $x \circ x^*$ is quasi-regular in $\mathfrak{A}$. A similar argument shows that $x^* \circ x$ is also quasi-regular in $\mathfrak{A}$. Therefore x must be quasi-regular in $\mathfrak{A}$, and the sufficiency is proved.

COROLLARY 1. *For every x in $\mathfrak{A}$,*

$$\sigma_{\mathfrak{A}}(x) = \sigma_{\mathfrak{B}}(x) \cup \overline{\sigma_{\mathfrak{B}}(x^*)}.$$

If x is normal,[8] *then $\sigma_{\mathfrak{A}}(x) = \sigma_{\mathfrak{B}}(x)$.*

COROLLARY 2. *If the involution in $\mathfrak{A}$ can be extended to an involution in $\mathfrak{B}$, then x is quasi-regular in $\mathfrak{A}$ if, and only if, x is quasi-regular in $\mathfrak{B}$. Hence, for every x in $\mathfrak{A}$, $\sigma_{\mathfrak{A}}(x) = \sigma_{\mathfrak{B}}(x)$.*

THEOREM 3. *For every x in $\mathfrak{A}$, it is true that*

(i) $\|x\|^2 \leqq \|xx^*\|_1 \leqq \|x\|_1\|x^*\|_1$.

(ii) $r_{\mathfrak{A}}(x) = r_{\mathfrak{B}}(x)$.

PROOF. By Corollary 1 of Theorem 2, $\sigma_{\mathfrak{A}}(xx^*) = \sigma_{\mathfrak{B}}(xx^*)$, so that $r_{\mathfrak{A}}(xx^*) = r_{\mathfrak{B}}(xx^*)$. Therefore

$$\|x\|^2 = \|xx^*\| = r_{\mathfrak{A}}(xx^*) = r_{\mathfrak{B}}(xx^*) \leqq \|xx^*\|_1 \leqq \|x\|_1\|x^*\|_1,$$

which proves (i). Now, for all n, we have $\|x^n\|^{2/n} \leqq \|x^n\|_1^{1/n}\|(x^*)^n\|_1^{1/n}$ and, on letting $n \to \infty$, obtain $r_{\mathfrak{A}}(x)^2 \leqq r_{\mathfrak{B}}(x) r_{\mathfrak{B}}(x^*)$. On the other hand, $r_{\mathfrak{A}}(x) = r_{\mathfrak{A}}(x^*)$ and always $r_{\mathfrak{B}}(x) \leqq r_{\mathfrak{A}}(x)$, since $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$. Therefore it follows that $r_{\mathfrak{A}}(x) = r_{\mathfrak{B}}(x)$.

COROLLARY. *Let $\mathfrak{A}$ be an arbitrary B^*-algebra and let $\|x\|_1$ be any norm under which $\mathfrak{A}$ is a (not necessarily complete) normed algebra. Then $r_{\mathfrak{A}}(x) \leqq \|x\|_1$ for every x. If, in addition, the second norm satisfies the condition $\|x^*\|_1 = \|x\|_1$, then $\|x\| \leqq \|x\|_1$ for every x. If the second norm satisfies the condition $\|xx^*\|_1 = \|x\|_1^2$, then*[9] *$\|x\| = \|x\|_1$ for every x.*

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[8] An element x is *normal* provided $xx^* = x^*x$. Observe that a normal element of $\mathfrak{A}$ can be contained in a maximal commutative B^*-subalgebra of $\mathfrak{A}$.

[9] This was proved by Kaplansky [5, Theorem 6.4] for B^*-algebras and is an extension of a similar result for C^*-algebras due to Gelfand and Neumark [2].

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