MONOTONIC SUBSEQUENCES

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1. Introduction. Hidden in a paper by Erdös and Szekeres is an intriguing result.

Basic Theorem. Every sequence \( S = \{x_i\} \) (\( i = 1 \) to \( n^2+1 \)) of real numbers having \( (n^2+1) \) terms possesses a (perhaps not strictly) monotonic subsequence \( M = \{x_j\} \) (\( j = 1 \) to \( n+1 \)) having \( (n+1) \) terms. Furthermore \( (n^2+1) \) is the smallest number for which this is true.

Briefly, this theorem states that a monotone subsequence of any desired length can be picked out from a sufficiently long sequence, and gives the precise lengths. An elegant proof of this theorem (unpublished) which is due to Martin D. Kruskal is sketched here.

Notation. Let \( S \) and \( T \) denote sequences, and let \( M \) and \( N \) denote monotone sequences. Let \( S(p) \), etc., denote a sequence having \( p \) terms. Let \( \psi(n) \) denote the least integer \( p \) such that every \( S(p) \) contains an \( M(n) \). In this notation we may restate the basic theorem thus:

Basic Theorem. For sequences of real numbers, \( \psi(n+1) = n^2+1 \).

To show that \( \psi(n+1) \geq n^2+1 \), it is sufficient to exhibit an \( S(n^2) \) which contains no \( M(n+1) \). Such a sequence is the following:

\[ n, \ldots, 1, 2n, \ldots, n+1, \ldots, n^2, \ldots, n^2 - n + 1. \]

To show that \( \psi(n+1) \leq n^2+1 \), assume the contrary and let \( n \) be the least integer such that \( \psi(n+1) > n^2+1 \). Let \( S(n^2+1) \) be a sequence which does not contain any \( M(n+1) \). Now define a majorant (minorant) of \( S \) to be a term which is strictly greater (smaller) than all terms following it in \( S \). The majorants (minorants) form a decreasing (increasing) subsequence of \( S \); hence there are at most \( n \) majorants and \( n \) minorants. As the final term of \( S \) is necessarily both a majorant and a minorant, there are at most \( 2n-1 \) extremants (majorants and minorants). The last term of every \( M(n) \) contained in \( S \) must be an extremant. Now delete from \( S \) all its extremants. The remainder \( S' \) can contain no \( M(n) \), yet has at least \( [(n^2+1)-(2n-1)] = [(n-1)^2+1] \) terms. Hence \( \psi(n) = \psi(n-1+1) > (n-1)^2+1 \). This contradicts the definition of \( n \) and completes the proof.

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2. First generalization—real vector spaces. In the following two sections the concept of a monotonic sequence is generalized to sequences of vectors from a finite-dimensional real vector space and a partial analogue of the Basic Theorem is obtained. (If \( S = \{ x_i \} \) is a sequence from a vector space, the subscript still distinguishes terms of the sequence, not components of a vector.)

Note that a sequence \( S = \{ x_i \} \) of real numbers is monotonic if and only if all the differences \((x_{i+1} - x_i)\) lie (perhaps not strictly) to the same side of 0 on the real line. This motivates:

**Definition.** A sequence \( S = \{ x_i \} \) of terms from a finite-dimensional real vector space is monotonic if there is a hyperplane \( H \) through the origin such that the differences \( d_i = x_{i+1} - x_i \) all lie in one of the closed half-spaces determined by \( H \).

This definition is further justified by:

**Lemma 1.** A sequence \( S \) of vectors is monotonic if and only if there is a directed line \( L \) such that the perpendicular projections of the \( x_i \) on \( L \) form an increasing sequence.

This lemma is easily proved by taking \( L \) and \( H \) to be perpendicular. The direction of any such line \( L \) is called a direction of monotonicity of \( S \). In 1-space there are only two possible directions of monotonicity: increasing and decreasing. In \( r \)-space the possible directions of monotonicity correspond to the points on the \((r-1)\)-sphere.

Any sequence of two real numbers is monotonic. This generalizes to:

**Lemma 2.** In \( r \)-space any sequence of \((r+1)\) terms is monotonic.

This can easily be proved geometrically.

**Notation.** For convenience, let the function \( \psi(n) \) which applies in \( r \)-space be indicated by \( \psi_r(n) \).

**Lemma 3.** (a) If \( S(n^2+1) = \{ x_i \} \) is any sequence of \((n^2+1)\) terms in \( r \)-space and \( L \) is any directed line in \( r \)-space, then \( S \) contains either a subsequence monotonic in the direction of \( L \) or a subsequence monotonic in the direction opposite to \( L \). (b) \( \psi_r(n+1) \leq n^2+1 \).

**Proof.** (a) follows easily from Lemma 1 and the Basic Theorem; (b) follows from (a). But (a) is much stronger than (b) because (a) says "for any \( L \cdots \)" while (b) says implicitly "there exists an \( L \) such that. . . ." This suggests that actually \( \psi_r(n+1) \) is smaller than \((n^2+1)\) in general.

What is the full generalization of the Basic Theorem to \( r \)-space? In other words, what is the function \( \psi_r(n) \)?
Conjecture. \( \psi(n+r) = r n + (n^2 - n + 1) \).
This conjecture is based solely on the following collection of facts.

**Basic Theorem.** \( \psi_1(n+1) = n + (n^2 - n + 1) \).

**Lemma 2 (New Form).** \( \psi_r(1+r) = r + (1-1+1) \).

**Theorem 1.** \( \psi_r(2+r) \leq 2r + 3 = 2r + (4-2+1) \).

**Lemma 4.** For \( r = 1 \) and 2, the \( \leq \) of Theorem 1 becomes =.

The proof of Theorem 1 is long and occupies the next section. For \( r = 1 \), Lemma 4 is trivial. To prove Lemma 4 for \( r = 2 \), it is sufficient to exhibit a sequence of 6 vectors from 2-space which contains no monotone subsequence of 4 terms. That the following is such a sequence may be verified directly:

\((2, -1), (3, 6), (-3, 12), (-3, -12), (3, -6), (2, 1)\).

3. **Proof of Theorem 1.** The basic tool in proving Theorem 1 is

**Lemma 5.** If \( S(p) = \{ x_i \} \) is a sequence in \( r \)-space, then at least one of the following conditions is true:
(a) \( S \) is monotone;
(b) there exist real numbers \( \alpha_i > 0 \) (\( i = 1, \ldots, p-1 \)) such that \( \sum \alpha_i d_i = 0 \), where \( d_i = x_{i+1} - x_i \).

**Proof.** It is sufficient to show that the falsity of (b) implies (a). Thus assume that 0 does not belong to the convex cone \( C = \{ \sum \alpha_i d_i \mid \text{all } \alpha_i > 0 \} \). Then a well known theorem about convex sets yields that there is a hyperplane \( H \) through 0 such that \( \overline{C} \) (the topological closure of \( C \)) lies entirely in one of the closed half-spaces determined by \( H \). Since \( d_i \) is in \( \overline{C} \) for all \( i \), \( S(p) \) is monotone, and the proof is complete.

**Comment.** It is possible to modify (b) into a necessary and sufficient condition for non-monotonicity. This condition might be useful in further investigation of the function \( \psi_r(n) \).

Lemma 6 follows from Lemma 5.

**Lemma 6.** Let

\( S(p) = \{ x_i \} \)

be any sequence in \( r \)-space, and let

\( S'(q) = \{ x_i \} \)

be a non-monotone subsequence of it (of course \( 1 \leq s_1 < s_2 < \cdots < s_q \leq p \)). Then there exists a \((p-1)\)-tuple of real numbers \( g = \{ g_i \} \) such
that $\sum \gamma_i d_s = 0$ where $g$ satisfies the following “suitability conditions with respect to $(s_1, \ldots, s_q)$”:

$$\begin{align*}
\begin{cases}
\gamma_1 = \cdots = \gamma_{s_1-1} = 0, \\
\gamma_{s_1} = \cdots = \gamma_{s_2-1} > 0, \\
\cdots \\
\gamma_{s_{q-1}} = \cdots = \gamma_{s_q-1} > 0, \\
\gamma_{s_q} = \cdots = \gamma_{p-1} = 0.
\end{cases}
\end{align*}$$

**Proof.** By Lemma 5 there exist strictly positive $\alpha_i$ ($i = 1, \ldots, q-1$) such that

$$\sum_{i=1}^{q-1} \alpha_i [x_{s_{i+1}} - x_i] = 0.$$ 

Hence

$$\sum_{i=1}^{q-1} \alpha_i \left[ \sum_{s_i}^{s_{i+1}-1} d_j \right] = 0.$$ 

Now define $g = \{\gamma_s\}$ as follows:

$$\begin{align*}
\begin{cases}
\gamma_1 = \cdots = \gamma_{s_1-1} = 0, \\
\gamma_{s_1} = \cdots = \gamma_{s_2-1} = \alpha_1 > 0, \\
\cdots \\
\gamma_{s_{q-1}} = \cdots = \gamma_{s_q-1} = \alpha_{q-1} > 0, \\
\gamma_{s_q} = \cdots = \gamma_{p-1} = 0.
\end{cases}
\end{align*}$$

Clearly $\sum \gamma_i d_s = 0$, and the proof is complete.

The structure of the $(p-1)$-tuple $g = (\gamma_1, \cdots, \gamma_{p-1})$ can be represented by a $q$-block diagram. This is obtained by substituting in $g$ an “$X$” for each nonzero $\gamma$, and “$0$” for each zero $\gamma$, and an “$=$” for each comma between two $\gamma$’s of one “block” of equal nonzero $\gamma$’s. A $g$ which is suitable with respect to $(3, 5, 6, 9)$ and which has 10 components is represented by the following 10-dimensional 4-block diagram: $(o, o, X=X, X, X=X=X, o, o)$. Block diagrams will be used extensively in the following arguments.

At this point it becomes necessary to consider the vectors of the fundamental $r$-space as $r$-tuples of real numbers. We shall write these $r$-tuples vertically and call them column vectors. We adopt the specific notation $d_s =$ the column vector $(\delta^s_t)$ as $t = 1, \cdots, r$, where $d_s$ has its usual significance. The sequence $\{d_s\}$, with $s = 1$ to
$p - 1$, now becomes a matrix $D = \|\delta_i\|$ in which $t = 1$ to $r$ is the row index and $s = 1$ to $p - 1$ is the column index. We shall let $d^t \ (t = 1, \ldots, r)$ represent the rows of $D$.

We now put Lemma 6 into the proper form for actual use:

**Lemma 7.** If $S(p) = \{x_i\}$ is a sequence in $r$-space, which does not contain any $M(q)$, then for each $(p - 1)$-dimensional $q$-block diagram there exists a $(p - 1)$-dimensional row vector $g$ such that $g$ is perpendicular to all the $d^t \ (t = 1, \ldots, r)$ and such that $g$ has the structure of the given $q$-block diagram.

**Proof.** The $q$-block diagram corresponds to a subsequence $S'(q) = \{x_i\}$ of $S$. Apply Lemma 6 to $S'(q)$ and rewrite the equation $\sum \gamma_i d_i = 0$ as $r$ equations $\sum \gamma_i d_i = 0$. These may be written $g \cdot d^t = 0$ or "$g$ is perpendicular to $d^t$." This completes the proof.

The last tool needed to prove Theorem 1 is

**Lemma 8.** If $S(p) = \{x_i\}$ is not monotonic, the vectors $d^t \ (t = 1 \to r)$ are linearly independent.

**Proof.** As $S$ is not monotonic, the $d_i$ do not all lie on a common hyperplane through the origin, hence span the whole $(r$-dimensional$)$ space of column vectors. Thus $D$ has column-rank $r$, hence row-rank $r$, which completes the proof.

Theorem 1 is proved indirectly. Assume contrary to the theorem that $S(2r + 3) = \{x_i\}$ contains no $M(r + 2)$.

By Lemma 8, the $r \ (2r + 2)$-dimensional row vectors $d^t$ are linearly independent. Now apply Lemma 7 to the following $(r + 2)$ different $(2r + 2)$-dimensional $(r + 2)$-block diagrams, and label the resulting $g$’s as shown:

<table>
<thead>
<tr>
<th></th>
<th>1 2 ··· $r + 1$ $r + 2$ $r + 3$ ··· $2r + 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g(1)$</td>
<td>$(X, X, \ldots, X, \ o, \ o, \ \ldots, \ o)$</td>
</tr>
<tr>
<td>$g(2)$</td>
<td>$(o, X, \ldots, X, X, \ o, \ \ldots, \ o)$</td>
</tr>
<tr>
<td></td>
<td>··· ··· ··· ··· ··· ··· ··· ··· ···</td>
</tr>
<tr>
<td>$g(r + 2)$</td>
<td>$(o, o, \ \ldots, \ o, X, X, \ \ldots, X)$</td>
</tr>
</tbody>
</table>

These $g$’s are called the fundamental $g$’s. Obviously the $(r + 2)$ fundamental $g$’s are linearly independent. By Lemma 7 every fundamental $g$ is perpendicular to every $d^t$. Therefore $\{g(1), \ldots, g(r + 2), d^1, \ldots, d^r\}$ is a basis for the $(2r + 2)$-space of row vectors. From this follows

**Lemma 9.** Every $g$ arising from application of Lemma 7 is a linear
combination of the fundamental g's.

At this point the proof of Theorem 1 splits into two cases, depending on whether \( r \) is odd or even; the former case is simpler and will be considered first.

Assume \( r \) is odd. Apply Lemma 7 to the following \((r+1)\) different \((2r+2)\)-dimensional \((r+2)\)-block diagrams and label the resulting g's as shown:

\[
\begin{array}{cccccccccccccccc}
1 & 2 & \cdots & r & r+1 & r+2 & r+3 & r+4 & \cdots & 2r+2 \\
\bar{g}(1) & (X, X, \cdots, X, X = X, o, o, \cdots, o) \\
\bar{g}(2) & (o, X, \cdots, X, X = X, X, o, \cdots, o) \\
& \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \\
\bar{g}(r+1) & (o, o, \cdots, o, X = X, X, X, \cdots, X).
\end{array}
\]

With the aid of Lemma 9 it is easy to see that

\[
\bar{g}(k) = \zeta(k)g(k) + \eta(k)g(k + 1)
\]

for properly chosen \( \zeta(k) > 0 \) and \( \eta(k) > 0 \). Introducing an obvious notation for the components of the g's and the \( \bar{g} \)'s, we have

\[
\gamma_{r+1}(k) = \zeta(k)\gamma_{r+1}(k) + \eta(k)\gamma_{r+1}(k + 1) = \gamma_{r+2}(k) = \zeta(k)\gamma_{r+2}(k) + \eta(k)\gamma_{r+2}(k + 1),
\]

which yields that

\[
[\gamma_{r+1}(k) - \gamma_{r+2}(k)] = -\epsilon(k)[\gamma_{r+1}(k + 1) - \gamma_{r+2}(k + 1)]
\]

where \( \epsilon(k) \) is a positive constant. Now

\[
\gamma_{r+1}(1) - \gamma_{r+2}(1) > 0
\]

because the first term is positive and the second term is 0. The preceding equation now yields successively

\[
\gamma_{r+1}(2) - \gamma_{r+2}(2) < 0, \quad \gamma_{r+1}(3) - \gamma_{r+2}(3) > 0,
\]

and so forth. Since \( r \) is odd, we obtain

\[
\gamma_{r+1}(r + 2) - \gamma_{r+2}(r + 2) = -\gamma_{r+2}(r + 2) > 0
\]

which is false. This completes the proof of Theorem 1 for odd values of \( r \).

Now assume that \( r \) is even. The following notation is introduced for convenience:

\[
\beta(k) \equiv \gamma_{r+1}(k) - \gamma_{r+2}(k), \quad \beta'(k) \equiv \gamma_{r+1}(k) - \gamma_{r+2}(k).
\]
Using the same method as in the preceding paragraph, the following inequalities are established (but without contradiction here):

\[
\beta(k) > 0 \quad \text{if } k \text{ is odd;}
\]

\[
\beta(k) < 0 \quad \text{if } k \text{ is even.}
\]

Now apply Lemma 7 to the following block diagrams and label the resulting g's as shown:

\[
\begin{array}{cccccccc}
g(1) & 1 & 2 & \cdots & r & r+1 & r+2 & r+3 & r+4 & r+5 & \cdots & 2r+2 \\
\end{array}
\]

\[
X, \ X, \ \cdots, \ X, \ \ X=\ X=\ X, \ \ o, \ \ o, \ \ \cdots, \ o
\]

\[
\begin{array}{cccccccc}
g(2) & 1 & 2 & \cdots & r & r+1 & r+2 & r+3 & r+4 & r+5 & \cdots & 2r+2 \\
\end{array}
\]

\[
X, \ X, \ \cdots, \ X, \ \ X=\ X=\ X, \ \ o, \ \ o, \ \ \cdots, \ o
\]

\[
\begin{array}{cccccccc}
g(r) & 1 & 2 & \cdots & r & r+1 & r+2 & r+3 & r+4 & r+5 & \cdots & 2r+2 \\
\end{array}
\]

\[
X=\ X=\ X, \ \ X, \ \ X, \ \ \cdots, \ X
\]

With the aid of Lemma 9 and the established inequalities for \(\beta(k)\), it is not difficult to show that

\[
g(k) = \xi(k)g(k) + \eta(k)g(k+1) + \theta(k)g(k+2)
\]

where \(\xi(k)\), \(\eta(k)\), and \(\theta(k)\) are positive constants. Translating these vector equations into component equations, and using the equalities among the components of the g's, we have

\[
\gamma_{r+1}(k) = \xi(k)\gamma_{r+1}(k) + \eta(k)\gamma_{r+1}(k+1) + \theta(k)\gamma_{r+1}(k+2)
\]

\[
= \gamma_{r+2}(k) = \xi(k)\gamma_{r+2}(k) + \eta(k)\gamma_{r+2}(k+1) + \theta(k)\gamma_{r+2}(k+2)
\]

\[
= \gamma_{r+3}(k) = \xi(k)\gamma_{r+3}(k) + \eta(k)\gamma_{r+3}(k+1) + \theta(k)\gamma_{r+3}(k+2).
\]

Subtract the second equation from the first, and then the third from the first:

\[
0 = \xi(k)\beta(k) + \eta(k)\beta(k+1) + \theta(k)\beta(k+2),
\]

\[
0 = \xi(k)\beta'(k) + \eta(k)\beta'(k+1) + \theta(k)\beta'(k+2).
\]

From these equations it follows that

\[
\xi(k) = \epsilon(k) \begin{pmatrix} \beta(k+1) & \beta(k+2) \\ \beta'(k+1) & \beta'(k+2) \end{pmatrix},
\]

\[
\eta(k) = \epsilon(k) \begin{pmatrix} \beta(k+2) & \beta(k) \\ \beta'(k+2) & \beta'(k) \end{pmatrix},
\]

\[
\theta(k) = \epsilon(k) \begin{pmatrix} \beta(k) & \beta(k+1) \\ \beta'(k) & \beta'(k+1) \end{pmatrix},
\]

where \(\epsilon(k)\) is a properly chosen constant of proportionality.
Call the three determinants $Z(k)$, $H(k)$, and $\Theta(k)$ respectively. Since $\xi(k)$, $\eta(k)$, and $\theta(k)$ are all positive, $Z(k)$, $H(k)$, and $\Theta(k)$ must all have the same sign as $\varepsilon(k)$. Furthermore, as $Z(k) = \Theta(k+1)$, all the determinants have the same sign for all $k$ (from 1 to $r$). To evaluate the sign of $\Theta(1)$ we use the already established inequalities for the $\beta$'s and find the sign of the $\beta$'s from direct examination of the block diagrams of the $g$'s. We see that

$$\Theta(1) = \begin{vmatrix} + & - \\ + & + \end{vmatrix} > 0,$$

so that all the determinants are positive.

Similarly, we find that

$$Z(1) = \begin{vmatrix} - & + \\ + & ? \end{vmatrix}.$$

For this to be positive, "?" must be "+", so that $\beta'(3) < 0$. Using this result we see that

$$Z(2) = \begin{vmatrix} + & - \\ - & ? \end{vmatrix}.$$

For this to be positive, "?" must be "+", so that $\beta'(4) > 0$. Similarly, $\beta'(5) < 0$, $\beta'(6) > 0$, and so forth. Since $r$ is even, $\beta'(r+2) > 0$; however, direct examination of the block diagram shows that $\beta'(r+2) < 0$. This contradiction completes the proof of Theorem 1 for even values of $r$, and hence the whole proof.

4. Second generalization—relation spaces. In the following sections we again generalize the Basic Theorem, but in a manner quite different from that of the preceding sections.

The Basic Theorem is not in essence a statement about the real number system. To see this, consider any set $X$ with an arbitrary binary relation $\subset$ over it. (No assumptions are made about $\subset$; for example, it need not be transitive.) Let us say that $S = \{x_i\}$ is $\subset$-monotonic ($\subset$-monotonic) if $x_i \subset x_{i+1}$ ($x_i \subset_{\subset} x_{i+1}$) for all $i$. Call $S$ monotonic if it is either $\subset$-monotonic or $\subset$-monotonic. Then for sequences over $X$ it is still true that $\psi(n+1) \leq n^2+1$, and for a "general" space $X$ it is true that $\psi(n+1) = n^2+1$. The inequality may be proved exactly as before.

What is the meaning of the "2" in $(n^2+1)$? The answer is simple: it is the number of relations ($\subset$ and $\subset$) of which at least one must hold between any two elements. The "2" is generalized to a "$k$" in Theorem 2.
Definition. A \( k \)-relation space (\( kR \)-space for short) consists of a set \( X \) and \( k \) binary relations \( C_h \) over \( X \) (\( h = 1, \ldots, k \)) satisfying one axiom: for every \( x, y \) in \( X \), there is at least one \( h \) depending on \( x \) and \( y \) such that \( x \in C_h y \).

Definition. A sequence \( S = \{ x_i \} \) is \( \subseteq_k \)-monotonic if \( x_i \subseteq_k x_{i+1} \) for all \( i \).

Definition. A sequence \( S \) is monotonic if there is at least one \( h \) for which it is \( \subseteq_k \)-monotonic.

Extended Basic Theorem. For sequences over a \( 2 \)-relation space, \( \psi(n+1) \leq n^2 + 1 \). Furthermore, there are \( 2 \)-relation spaces for which \( \psi(n+1) = n^2 + 1 \).

Theorem 2. For sequences over a \( kR \)-space, \( \psi(n+1) \leq n^k + 1 \). Furthermore, there are \( kR \)-spaces for which \( \psi(n+1) = n^k + 1 \).

In the \( kR \)-space to be described \( \psi(n+1) = n^k + 1 \). Let \( X \) consist of all real polynomials in the variable \( \xi \) of degree \( \leq k - 1 \). The relations \( C_h \) are defined by

\[
p(\xi) \subseteq_k q(\xi) \quad \text{if} \quad [p(\xi) - q(\xi)] \text{ has exactly degree } (k - h).
\]

(The zero polynomial is assigned degree 0.) It is trivial to show that this is a \( kR \)-space, and the following \( S(n^k) \) contains no \( M(n+1) \):

\[
\begin{align*}
\xi + 1, \\
\xi^2 + \xi + 1, \\
\xi^3 + \xi^2 + \xi + 1, \\
\cdots, \\
\xi^{k-1} + \xi^{k-2} + \cdots + \xi + n, \\
\xi^{k-1} + \xi^{k-2} + \cdots + 2\xi + 1, \\
\cdots, \\
\xi^{k-1} + \xi^{k-2} + \cdots + 2\xi + n, \\
\cdots, \\
n\xi^{k-1} + n\xi^{k-2} + \cdots + n\xi + n.
\end{align*}
\]

The proof\(^\star\) that \( \psi(n+1) \leq n^k + 1 \) in a \( k \)-relation space rests on Lemma 10 which (for real numbers) is stated in the paper by Erdős and Szekeres.

\(^\star\) For the basic idea of this proof I am indebted to the referee, who suggested a proof far simpler than the one originally contained in my paper. However, by using Lemma 10, not originally in my paper and not known to the referee, I have further simplified his proof.
Lemma 10. If \((X, \subseteq_1, \subseteq_2)\) is a 2-relation space, then any sequence \(S(p+1)\) either contains a \(\subseteq_1\)-monotonic subsequence \(M(p+1)\) or a \(\subseteq_2\)-monotonic subsequence \(M(q+1)\).

This lemma may easily be proved in the same way as the Extended Basic Theorem.

Now we proceed by an induction on \(A\). If \((X, \subseteq_1, \ldots, \subseteq_{t+1})\) is a \((k+1)\)-relation space, and \(S(n^{k+1}+1)\) is a sequence over it, define \(\ll_1\) and \(\ll_2\) by

\[
\begin{align*}
x \ll_1 y, & \quad \text{if } x \subseteq_h y \text{ for any } h \text{ from 1 to } k, \\
x \ll_2 y, & \quad \text{if } x \subseteq_{k+1} y.
\end{align*}
\]

Now \((X, \ll_1, \ll_2)\) is a 2-relation space. Hence by Lemma 10, \(S\) contains either \(M_1(n^k+1)\) which is \(\ll_1\)-monotonic or \(M_2(n+1)\) which is \(\ll_2\)-monotonic. In the latter case the proof is complete as \(M_2(n+1)\) is also \(\subseteq_{k+1}\)-monotonic. In the former case, let \(M_1\) denote the set of elements in \(\subseteq_1(\subseteq^{k+1})\) and define \(\subseteq_h\) over \(M_1\) by

\[
x \subseteq_h y \quad \text{if } x \subseteq_h y \text{ or if } y \text{ precedes } x \text{ in } \subseteq_1(n^{k+1}).
\]

Then \((M_1, \subseteq_1, \ldots, \subseteq^k)\) is a \(kR\)-space. Hence by the induction hypothesis \(M_1(n^k+1)\) must contain an \(M(n+1)\) which is \(\subseteq_h\)-monotonic for some \(h\) from 1 to \(k\). But then \(M(n+1)\) is \(\subseteq_h\)-monotonic, which completes the proof.

5. De Bruijn's Theorem—a generalization. In some unpublished work N. G. de Bruijn has generalized the Basic Theorem to sequences of \(m\)-tuples of real numbers. He defines such a sequence to be monotonic if each component sequence is monotonic. (Thus there are \(2^m\) "directions" of monotonicity.)

De Bruijn's Theorem. Over the space of \(m\)-tuples, \(\psi(n+1) = n^{2^m}+1\).

His proof is simply an \(m\)-fold application of the Basic Theorem. From \(S(n^{2^m}+1)\) pick a subsequence \(S_1(n^{2^m-1}+1)\) whose first components are monotonic. From \(S_1\) pick a subsequence \(S_2(n^{2^m-2}+1)\) whose second components are monotonic; and so forth. This eventually yields \(S_m(n+1)\) which is monotonic. This shows that \(\psi(n+1) \leq n^{2^m}+1\); the opposite inequality is easily verified.

De Bruijn's Theorem inspires Theorem 3, which is at once a generalization of De Bruijn's Theorem and of Theorem 2.

Definition. A joint relation-space with coefficients \(k_1, \ldots, k_m\) consists of \(m\) different relation-spaces over the same set \(X\) such that the \(i\)th space is a \(k_iR\)-space.
The relations are denoted by $\subset^{l}_k$, where $k = 1, \ldots, k_1$ and $l = 1, \ldots, m$.

**Definition.** A sequence is monotonic over a joint relation-space if it is simultaneously monotonic over every one of the $k_1R$-spaces.

**Theorem 3.** Over a joint relation-space, $\psi(n + 1) \leq n^{k_1 \cdots k_m} + 1$.

De Bruijn's Theorem is a special case of Theorem 3 in which all the coefficients are 2 and the set $X$ consists of the real $m$-tuples. However his proof cannot be extended to prove Theorem 3, for his proof depends on the transitivity of his relations which is not assumed in Theorem 3.

However Theorem 3 may be proved as a trivial corollary to Theorem 2. Simply define $k_1k_2 \cdots k_m$ new relations over $X$ by

$$x \ll_{k_1, \ldots, k_m} y \text{ only if } x \subset^{l}_k y \text{ for all } l.$$  

Then $X$ and the new relations form a $k_1k_2 \cdots k_mR$-space. Use of Theorem 2 then completes the proof.

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