NOTE ON DIRICHLET SERIES. IV. ON THE SINGULARITIES OF DIRICHLET SERIES

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Let us put

$$(1) \qquad F(s) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n s)$$

$$(s = \sigma + it,\ 0 \leqq \lambda_1 < \lambda_2 < \cdots < \lambda_n \to +\infty).$$

When we vary coefficients $\{a_n\}$, this change has some influence upon singularities. Concerning this problem, O. Szász [1, p. 107] has proved the next theorem, which is a generalization of Hurwitz-Pólya's theorem [2, p. 36]:

O. Szász's Theorem. *Let* (1) *have the finite simple convergence-abscissa* σ_s. *If* $\lim_{n\to\infty} \log n/\lambda_n = 0$, *then there exists a sequence* $\{\epsilon_n\}$ $(\epsilon_n = \pm 1)$ *such that* $\sum_{n=1}^{\infty} a_n\epsilon_n \exp(-\lambda_n s)$ *has* $\sigma = \sigma_s$ *as the natural boundary.*

In this note, we shall prove the following theorem of the same type:

Theorem. *Let* (1) *have the finite simple convergence-abscissa* σ_s. *If* $\lim_{n\to\infty} \log n/\lambda_n = 0$, *then there exists a Dirichlet series* $\sum_{n=1}^{\infty} b_n \exp(-\lambda_n s)$ *having* $\sigma = \sigma_s$ *as the natural boundary such that*

(a) $|b_n| = |a_n|$ $(n = 1, 2, \cdots)$, *and* $\lim_{n\to\infty} |\arg(a_n) - \arg(b_n)| = 0$

or

(b) $\arg(b_n) = \arg(a_n)$ $(n = 1, 2, \cdots)$, *and* $\lim_{n\to\infty} |b_n/a_n| = 1$.

Proof. On account of $\lim_{n\to\infty} \log n/\lambda_n = 0$, and G. Valiron's theorem [3, p. 4], we get

$$(2) \qquad \sigma_s = \limsup_{n\to\infty} 1/\lambda_n \cdot \log |a_n|.$$

Therefore we can select a subsequence $\{\lambda_{n_i}\}$ such that

$$(3) \quad \begin{aligned} &\text{(i)} \qquad \sigma_s = \lim_{n\to\infty} 1/\lambda_{n_i} \cdot \log |a_{n_i}|, \\ &\text{(ii)} \qquad \liminf_{i\to\infty} (\lambda_{n_i+1} - \lambda_{n_i}) > 0, \qquad \lim_{i\to\infty} i/\lambda_{n_i} = 0. \end{aligned}$$

Again by G. Valiron's theorem and (3) (i), $G_1(s;\ \theta,\ \alpha)$

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$=\sum_{i=1}^{\infty} a_{n_i} \exp(\alpha\theta/\lambda_{n_i})\times\exp(-\lambda_{n_i}s)$ has the simple convergence-abscissa σ_s, where (i) θ is a real constant, (ii) α is a constant determined later. Hence $G_2(s)=\sum_{n\notin\{n_i\}} a_n \exp(-\lambda_n s)$ is simply convergent at least for $\sigma>\sigma_s$. Now let us put

$$F(s;\theta,\alpha) = G_1(s;\theta,\alpha)+G_2(s), \tag{4}$$

which is evidently simply convergent at least for $\sigma>\sigma_s$.

Denote by $E(\theta,\alpha)$ the set of regular points of $F(s;\theta,\alpha)$ on $\sigma=\sigma_s$, which is clearly an open set. Then we can prove that

$$E(\theta_1,\alpha)\cap E(\theta_2,\alpha) = 0 \qquad \text{for } \theta_1\neq\theta_2. \tag{5}$$

In fact, if there should exist a point ξ on $\sigma=\sigma_s$ such that $\xi\in E(\theta_1,\alpha)\cap E(\theta_2,\alpha)\neq 0$, then $F(s;\theta_1,\alpha)-F(s;\theta_2,\alpha)$ would be regular at $s=\xi$. On the other hand, since

$$\begin{aligned} F(s;\theta_1,\alpha)-F(s;\theta_2,\alpha) &= \sum_{i=1}^{\infty} a_{n_i}\{\exp(\alpha\theta_1/\lambda_{n_i})-\exp(\alpha\theta_2/\lambda_{n_i})\}\exp(-\lambda_{n_i}s) \\ &= \sum_{i=1}^{\infty} a_{n_i}O(1/\lambda_{n_i})\exp(-\lambda_{n_i}s), \end{aligned}$$

taking account of (3), G. Valiron's theorem, and Carlson-Landau's theorem [3, pp. 140–141], $F(s;\theta_1,\alpha)-F(s;\theta_2,\alpha)$ has the simple convergence-abscissa σ_s, and furthermore it has $\sigma=\sigma_s$ as the natural boundary, which contradicts the regularity at $s=\xi$. Hence, (5) holds.

If $E(\theta,\alpha)\neq 0$ should hold for all θ, $0\leqq\theta\leqq\gamma$ (γ a fixed constant), then, by (5), $\{F(s;\theta,\alpha)\}$ is at most of enumerable power, which contradicts the power of continuum of $\{F(s;\theta,\alpha)\}$. Hence, for at least one θ', $E(\theta',\alpha)=0$ holds. If we put $\alpha=(-1)^{1/2}(-1)$, then (a) ((b)) is valid. q.e.d.

References

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