ON INSCRIBING n-DIMENSIONAL SETS IN A REGULAR n-SIMPLEX

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1. Introduction. It was proved in 1901 [1] that every set of diameter 1 in n-space can be inscribed in an n-sphere of radius $r \leqq (n/2(n+1))^{1/2}$. The best and most recent proof was given in 1941 [2]. The geometric content of this result is made apparent by the following equivalent formulation: the circumscribed n-sphere of a regular n-simplex of diameter 1 will cover any n-dimensional set of diameter 1.

In this paper we prove a dual to the above result, namely (Theorem 1): the regular n-simplex whose inscribed sphere has diameter 1 will cover any n-dimensional set of diameter 1. We then apply this result to the case $n=2$ to show that every plane set of diameter 1 is the union of three sets each of diameter less than $3^{1/2}/2$. This result is related to the still unsolved conjecture of K. Borsuk [3] that every set in n-space of diameter 1 is the union of $n+1$ sets each of diameter less than 1. Our result gives the strongest possible answer to this question for $n=2$.

2. Main theorem.

Notation. Throughout this paper lower case letters will denote vectors, Greek letters will denote (real) scalars, the scalar product of the vectors x and y will be denoted by xy, and the norm of a vector x by $|x|$.

DEFINITIONS. By an *n-simplex* Σ in a Euclidean space we shall mean the convex closure of a set of $n+1$ vectors $v_0, v_1, \cdots, v_n$. We shall say that Σ is *regular* if the numbers $|v_i - v_j|$ are equal for $i \neq j$, $i, j = 0, \cdots, n$. An $(n-1)$*-face* of Σ is the convex hull of any n vectors from among the v_i. A set S is said to be *inscribed* in Σ if $S \subset \Sigma$ and S intersects every $(n-1)$-face of Σ.

We now state the principal result.

THEOREM I. *Let S be a closed subset of n-space of diameter* 1. *Then S can be inscribed in a regular n-simplex of diameter $d \leqq (n(n+1)/2)^{1/2}$.*

PROOF. Let V denote $(n+1)$-dimensional Euclidean space with unit vectors $e_0, e_1, \cdots, e_n$. Let $\bar{e} = \sum_{i=0}^{n} e_i$ and let $L = \{x \mid x \in V, x\bar{e} = 0\}$. Since L is an n-dimensional linear subspace of V we may assume that S is imbedded in L.

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Let $\alpha_i = \min_{x\in S} xe_i$, and let $a = \sum_{i=0}^n \alpha_i e_i$.
Let $\beta_i = \max_{x\in S} xe_i$, and let $b = \sum_{i=0}^n \beta_i e_i$.
Let $H_i^- = \{x \mid xe_i \geqq \alpha_i\}$, $H_i^+ = \{x \mid xe_i \leqq \beta_i\}$.
Let $\Sigma^- = \bigcap_{i=0}^n H_i^- \cap L$, $\Sigma^+ = \bigcap_{i=0}^n H_i^+ \cap L$.

We shall show that Σ^- and Σ^+ are regular n-simplexes and at least one of them has diameter $d \leqq (n(n+1)/2)^{1/2}$. The proof will be divided into three parts.

(1) We first show that $a\bar{e} \leqq 0$ and $a\bar{e} = 0$ only if S consists of a single point.

For any $x \in S$, $xe_i \geqq \alpha_i$, hence $\sum_{i=0}^n (xe_i) = x\bar{e} \geqq \sum_{i=0}^n \alpha_i = a\bar{e}$, but $x \in L$ so $x\bar{e} = 0$, hence $a\bar{e} \leqq 0$. If $a\bar{e} = 0$ then $\sum_{i=0}^n \alpha_i = 0$ and for any $x \in S$, $\sum_{i=0}^n xe_i = 0$, but $xe_i - \alpha_i \geqq 0$, so $\sum_{i=0}^n (xe_i - \alpha_i) = 0$ implies $xe_i = \alpha_i$ for all i, hence $x = a$. Similarly one shows that $b\bar{e} \geqq 0$ and equality holds only if S consists of a single point. Since in the latter case the theorem is trivial we assume henceforth that $a\bar{e} < 0$ and $b\bar{e} > 0$.

(2) Σ^- and Σ^+ are n-simplexes and S is inscribed in both.

We shall show that Σ^- is the simplex spanned by the vectors $v_0, v_1, \cdots, v_n$ where $v_i = a - (a\bar{e})e_i$. First suppose $x = \sum_{i=0}^n \lambda_i v_i$ where $\lambda_i \geqq 0$ and $\sum_{i=0}^n \lambda_i = 1$. Then $x = \sum_{i=0}^n \lambda_i(a - (a\bar{e})e_i) = a - (a\bar{e}) \sum_{i=0}^n \lambda_i e_i$, so $xe_i = \alpha_i - \lambda_i(a\bar{e}) \geqq \alpha_i$ since $a\bar{e} < 0$. Also $x\bar{e} = \sum_{i=0}^n xe_i = \sum_{i=0}^n \alpha_i - a\bar{e} = 0$, so $x \in \Sigma^-$.

On the other hand suppose $x \in \Sigma^-$. Then $x = \sum_{i=0}^n \mu_i e_i$ where $\mu_i = xe_i \geqq \alpha_i$ and $\sum_{i=0}^n \mu_i = 0$. Now since $a\bar{e} \neq 0$ we can write $e_i = (1/a\bar{e})(a - v_i)$, so $x = (1/a\bar{e}) \sum_{i=0}^n \mu_i(a - v_i) = -(1/a\bar{e}) \sum_{i=0}^n \mu_i v_i$. From the definition of v_i we see that $\sum_{i=0}^n \alpha_i v_i = 0$, so we may write

$$(*) \qquad x = -(1/a\bar{e}) \sum_{i=0}^n \mu_i v_i + (1/a\bar{e}) \sum_{i=0}^n \alpha_i v_i = (1/a\bar{e}) \sum_{i=0}^n (\alpha_i - \mu_i) v_i,$$

and $(\alpha_i - \mu_i)/a\bar{e} \geqq 0$ since $\mu_i \geqq \alpha_i$ and $a\bar{e} < 0$. Also $(1/a\bar{e}) \sum_{i=0}^n (\alpha_i - \mu_i) = \sum_{i=0}^n \alpha_i / a\bar{e} = 1$. Thus Σ^- is the convex closure of the v_i's.

To show that S is inscribed in Σ^- we must show that it has a point in common with each $(n-1)$-face of Σ^-. Now there is an $x \in S$ such that $xe_i = \mu_i = \alpha_i$. Using this in the expression (*) above we see that x lies in the $(n-1)$-face of Σ^- opposite the vertex v_i.

The proof for Σ^+ is similar.

(3) Σ^- and Σ^+ are regular simplexes of diameter $d^- = -2^{1/2} a\bar{e}$ and $d^+ = 2^{1/2} b\bar{e}$ respectively, and min $(d^-, d^+) \leqq (n(n+1)/2)^{1/2}$.

Working with Σ^-, for $i \neq j$ we have $v_i - v_j = -a\bar{e}(e_i - e_j)$ so $|v_i - v_j| = ((ae)^2(e_i - e_j)^2)^{1/2} = 2^{1/2}|a\bar{e}| = -2^{1/2} a\bar{e} = d^-$, and similarly each edge of Σ^+ has length $d^+ = 2^{1/2} b\bar{e}$. To estimate min (d^-, d^+) we first show that $\beta_i - \alpha_i \leqq (n/(n+1))^{1/2}$ for all i. It suffices to show this for $\beta_0 - \alpha_0$. Choose x and $y \in S$ such that $xe_0 = \alpha_0$, $ye_0 = \beta_0$ and let $x = \alpha_0 e_0$

$+\sum_{i=1}^n \gamma_i e_i$, $y=\beta_0 e_0+\sum_{i=1}^n \delta_i e_i$ and let $\theta_0=\beta_0-\alpha_0$, $\theta_i=\delta_i-\gamma_i$ for $i>0$. Then $\sum_{i=0}^n \theta_i=0$ and $\sum_{i=0}^n \theta_i^2\leqq 1$, using for the first time the hypothesis that the diameter of S is 1. Now $0\leqq \sum_{i=1}^n (\theta_i+\theta_0/n)^2 = \sum_{i=1}^n \theta_i^2+(2\theta_0/n)\sum_{i=1}^n \theta_i+\theta_0^2/n=\sum_{i=1}^n \theta_i^2-\theta_0^2/n$, so $\theta_0^2/n\leqq \sum_{i=1}^n \theta_i^2$ and $((n+1)/n)\theta_0^2\leqq \sum_{i=0}^n \theta_i^2\leqq 1$ which gives $\theta_0\leqq (n/(n+1))^{1/2}$.

Now $d^+ + d^- = 2^{1/2}(b\bar{e}-a\bar{e})=2^{1/2}\sum_{i=0}^n (\beta_i-\alpha_i)\leqq (2n(n+1))^{1/2}$ and since d^+ and d^- are both positive it follows that one of them is less than or equal to $(n(n+1)/2)^{1/2}$ completing the proof.

REMARK. One easily computes the diameter of the regular n-simplex whose inscribed sphere has diameter 1, and finds it to be exactly $(n(n+1)/2)^{1/2}$. Therefore the inequality we have obtained is the best possible.

3. **Applications.** We use the preceding result to prove the following theorem.

THEOREM II. *Every plane set S of diameter 1 is the union of 3 sets each of diameter $d\leqq 3^{1/2}/2$.*

PROOF. By Theorem I, S can be inscribed in an equilateral triangle Δ of side $s\leqq 3^{1/2}$. Assuming the most unfavorable case, i.e., that $s=3^{1/2}$, we shall describe the desired decomposition.

Let Δ have vertices A, B, and C, let a and b be the midpoints of BC and AC respectively, and let O be the center of Δ. Let γ be a line parallel to and of distance 1 from AB, intersecting AC and BC in points p and q respectively. Since the diameter of S is 1, S lies between lines γ and AB. Let R_1 be the pentagon with vertices $O\ b\ p\ q\ a$ and let R_2 and R_3 be the region obtained by rotating R_1 about O through angles $2\pi/3$ and $4\pi/3$. Then $S\subset R_1\cup R_2\cup R_3$. By direct computation one shows that the diameter of R_1 is the distance from a to b which is $3^{1/2}/2$, and since R_2 and R_3 are congruent to R_1 the theorem is proved.

REMARK. It is easily shown that a disc of diameter 1 cannot be decomposed into three sets each having diameter less than $3^{1/2}/2$ and therefore the inequality of Theorem I is the best possible. This result suggests a strengthening of the Borsuk conjecture, as follows. Let the n-sphere of diameter 1 be decomposed into $n+1$ sets, $S_0, S_1, \cdots, S_n$ so that $\max_i$ (diameter S_i) $=d_n$ where d_n is as small as possible. Then every set of diameter 1 in n-space is the union of $n+1$ sets of diameter at most equal to d_n.

As a further application of the main theorem one can prove the known fact [4] that every plane set of diameter 1 can be imbedded in a regular hexagon the distance between whose opposite sides is 1.

In three-space one can show that every set of diameter 1 can be imbedded in a regular octahedron the distance between whose opposite faces is 1. We omit the proofs.

Bibliography

1. H. W. E. Jung, *Über die kleinste kugel, die eine räumliche Figur einschliesst,* J. Reine Angew. Math. vol. 123 (1901) pp. 241–257.

2. L. M. Blumenthal and G. E. Wahlin, *On the spherical surface of smallest radius enclosing a bounded subset of n-dimensional Euclidean space,* Bull. Amer. Math. Soc. vol. 47 (1941) pp. 771–777.

3. K. Borsuk, *Drei Sätze über die n-dimensionale euklidische Sphäre,* Fund. Math. vol. 20 (1933) pp. 177–190.

4. T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper,* Springer, 1934, pp. 130–131.

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