CONTINUA WHICH ARE THE SUM OF A FINITE NUMBER OF INDECOMPOSABLE CONTINUA

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Swingle [7] has given the following definitions. (1) A continuum $M$ is said to be the finished sum of the continua of a collection $G$ if $G^* = M$ and no continuum of $G$ is a subset of the sum of the others. (2) If $n$ is a positive integer, the continuum $M$ is said to be indecomposable under index $n$ if $M$ is the finished sum of $n$ continua and is not the finished sum of $n+1$ continua.

Swingle has shown [7, Theorem 2] that if $n$ is a positive integer and the continuum $M$ is indecomposable under index $n$, then $M$ is the finished sum of $n$ indecomposable continua. The author has shown [2, Theorem 1] that if $n=2$ and the continuum $M$ is indecomposable under index $n$, and $G$ is a collection of $n$ indecomposable continua whose finished sum is $M$, then $G$ is the only such collection.

In the present paper, it is shown that for a compact continuum, this theorem holds for any positive integer $n$. Also, there is given a necessary and sufficient condition that a compact continuum be indecomposable under index $n$.

An indecomposable continuum can be described as a nondegenerate continuum which is indecomposable under index 1. If $n=1$, then in order that a continuum $M$ be indecomposable under index $n$, it is necessary and sufficient that $M$ contain $n+2$ points such that $M$ is irreducible about any $n+1$ of them. Swingle [7] has shown that it is impossible, in a certain manner, to generalize this theorem. Theorem 3 of the present paper might be considered a generalization of the necessary condition of the above theorem. However, it is easily seen that the converse of Theorem 3 is not true.

Theorems 1–5 are proved on the basis of R. L. Moore's Axioms 0 and 1. Hence these theorems hold in any metric space.

Theorem 1. If $n>1$ and the compact continuum $M$ is the sum of $n$ indecomposable continua $M_1, M_2, \ldots, M_n$ such that, for each $i (i \leq n)$, a composant $K_i$ of $M_i$ does not intersect $M_1 + M_2 + \cdots + M_{i-1}$

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1 Numbers in brackets refer to the bibliography at the end of this paper.

2 The sum of the continua of $G$ is denoted by $G^*$.

3 For a proof of this theorem, see [4, Theorem IV].

4 Moore's axioms are stated in [5]. The first three parts of Axiom 1 are denoted by Axiom 1c.

4 If $P$ is a point of a continuum $M$, the set of all points $X$ such that $P + X$ lies in a proper subcontinuum of $M$ is called a composant of $M$. 

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+ M_{i+1} + \cdots + M_n, then \( M \) is indecomposable under index \( n \).

**Proof.** Suppose that there is a collection \( G \) consisting of \( n+1 \) continua whose finished sum is \( M \). No continuum of \( G \) is a proper subset of one of the indecomposable continua \( M_1, M_2, \cdots, M_n \). Hence, for each \( i \) (\( i \leq n \)), if \( K_i \) intersects a continuum \( X \) of \( G \), then \( X \) contains \( M_i \). Consequently, there exist \( n \) continua of \( G \) such that their sum is \( M \). This is contrary to the supposition that \( M \) is the finished sum of the continua of \( G \). Since \( M \) is the finished sum of the continua \( M_1, M_2, \cdots, M_n \), then it is indecomposable under index \( n \).

**Theorem 2.** If \( n \) is a positive integer and the compact continuum \( M \) is indecomposable under index \( n \), then there is only one collection of indecomposable continua whose finished sum is \( M \).

**Proof.** By [7, Theorem 2], there is a collection \( G \) consisting of \( n \) indecomposable continua \( M_1, M_2, \cdots, M_n \) such that \( M \) is their finished sum. By [3, Theorem 1], for each \( i \) (\( i \leq n \)), some composant \( K_i \) of \( M_i \) does not intersect \((G-M_i)^*\). Suppose that there is a collection \( G' \) of indecomposable continua such that \( G' \neq G \) and \( M \) is the finished sum of the continua of \( G' \). Let \( i \) be a positive integer not greater than \( n \). Some continuum \( X_i \) of \( G' \) intersects \( K_i \). Neither of the indecomposable continua \( X_i \) and \( M_i \) is a proper subset of the other. Since no proper subcontinuum of \( M_i \) intersects both \( K_i \) and \((G-M_i)^*\), then \( X_i \subseteq M_i \). Hence \( G' = G \).

**Theorem 3.** If \( n > 1 \) and the compact continuum \( M \) is indecomposable under index \( n \), then there is a subset \( H \) of \( M \) consisting of \( 2n \) points such that \( M \) is irreducible about every subset of \( H \) consisting of \( 2n-1 \) points.

**Proof.** Let \( M_1, M_2, \cdots, M_n \) be \( n \) indecomposable continua whose finished sum is \( M \). For each \( i \) (\( i \leq n \)), let \( K_i \) be a composant of \( M_i \) as described in the proof of Theorem 2. There exists a subset \( H \) of \( M \) such that for each \( i \) (\( i \leq n \)), \( H \cdot M_i \) consists of two points of \( K_i \). The set \( H \) satisfies the requirements of the conclusion of Theorem 3.

**Theorem 4.** If \( n > 1 \), \( M \) is a compact continuum, \( G \) is a collection consisting of \( n \) indecomposable continua whose finished sum is \( M \), and \( H \) is a finite set of points about which \( M \) is irreducible, then \( M \) is indecomposable under index \( n \).

**Lemma 4.1.** If the hypothesis of Theorem 4 is satisfied, \( X \) is a continuum of \( G \), and \( T \) is a component of \((G-X)^*\), then some composant of \( X \) does not intersect \( T \).
Proof of Lemma 4.1. Suppose that every composant of $X$ intersects $T$. Then there exists a finite collection $W$ of proper subcontinua of $X$ such that $W^*+(G-X)^*$ is connected. There exists a finite collection $Y$ of proper subcontinua of $X$ such that (1) every continuum of $Y$ intersects $(G-X)^*$ and (2) if $X$ intersects $H$, then $Y^*$ contains $X\cdot H$. Since $X$ is indecomposable and $M$ is the finished sum of the continua of $G$, then $Y^*+W^*$ does not contain $M-(G-X)^*$. Therefore, $W^*+Y^+(G-X)^*$ is a proper subcontinuum of $M$ containing $H$. This is a contradiction since $M$ is irreducible about $H$.

Proof of Theorem 4. An inductive argument will be used. Suppose that Theorem 4 is not true. Let $k$ be the smallest positive integer $n$ such that if $M$ is a compact continuum satisfying the hypothesis of Theorem 4, then $M$ is not indecomposable under index $n$. By Theorem 1, there is a continuum $X$ of $G$ such that every composant of $X$ intersects $(G-X)^*$. By Lemma 4.1, $(G-X)^*$ is not connected. Therefore, $k>2$. The set $(G-X)^*$ is the sum of a finite number of mutually exclusive continua. Let $T$ be one of these continua. Since $M$ is irreducible about $H$, then $T\cdot T-X$ contains a point of $H$. By Lemma 4.1, there is a composant of $X$ which does not intersect $T$. Let $P$ be a point of such a composant. The continuum $T+X$ is irreducible about the finite set $H\cdot T+P$. There is a positive integer $j$ less than $k$ such that $T+X$ is the finished sum of $j$ continua of $G$. Then $T+X$ is indecomposable under index $j$. By [3, Theorem 1], every continuum of $G$ which is a subset of $T+X$ contains a composant which does not intersect any other continuum of $G$ which is a subset of $T+X$. Therefore, every continuum of $G-X$ contains a composant which does not intersect any other continuum of $G$. Let $L$ be a collection consisting of $k-1$ points such that if $Z$ is a continuum of $G-X$, then a point of $L$ belongs to a composant of $Z$ lying in $M-(G-Z)^*$. Since, by supposition, $M$ is not indecomposable under index $k$, then there is a collection $G'$ consisting of $k+1$ continua whose finished sum is $M$. Since the set $L$ is contained in the sum of $k-1$ continua of $G'$, then $(G-X)^*$ is contained in the sum of $k-1$ continua of $G'$. Hence there exist two continua $X_1$ and $X_2$ of $G'$ such that each of them contains a point of $M-(G-X)^*$ which does not belong to any other continuum of $G'$. Let $R$ be a domain intersecting $X_1$ and not intersecting $(G'-X_1)^*+(G-X)^*$. Every composant of $X$ intersects $R$. Therefore, there exists a finite collection $W$ of proper sub-

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* This follows from the fact that every proper subcontinuum of an indecomposable continuum $M$ is a continuum of condensation of $M$ [4, Theorem III] and the fact that no indecomposable continuum is the sum of a finite number of its proper subcontinua [4, Theorem III].
continua of $X$ such that $X_1 + W^* + (G - X)^*$ is a continuum. Let $Y$ be a finite collection of continua as described in the proof of Lemma 4.1. Since $X_1 + Y^* + W^* + (G - X)^*$ is a subcontinuum of $M$ containing $H$, then $X_1 + Y^* + W^* + (G - X)^* = M$. Since $X$ is indecomposable and $X_1 + (G - X)^*$ contains $X - (Y^* + W^*)$, then $X_1 + (G - X)^*$ contains $X$. This is impossible since $X_1 + (G - X)^*$ does not contain $X_2$. Thus the supposition that Theorem 4 is not true has led to a contradiction.

Theorem 5. If $n > 1$, then in order that the compact continuum $M$ should be indecomposable under index $n$, it is necessary and sufficient that $M$ should be the finished sum of $n$ indecomposable continua and be irreducible about some $n$ points.

The necessity follows from [7, Theorem 2] and [3, Theorem 2]. The sufficiency follows from Theorem 4.

Theorem 6. If the compact continuum $M$ in the plane is the finished sum of two indecomposable continua $H$ and $K$ such that some composant of $H$ does not intersect $K$, then $M$ is indecomposable under index two.

Lemma 6.1. If the hypothesis of Theorem 6 is satisfied and $K_1$ and $K_2$ are mutually exclusive simple discs intersecting $K$ but not $H$, then there do not exist four mutually exclusive continua $W_1$, $W_2$, $W_3$, and $W_4$ such that, for each $i (i \leq 4)$, $W_i$ belongs to $K$, intersects $H$, and is irreducible from $K_1$ to $K_2$.

Proof of Lemma 6.1. Suppose that there do exist four such continua. Let $D$ denote the complementary domain of $K_1 + K_2$. Consider the case in which $W_3 + W_4$ separates $W_1$ from $W_2$ in $D$. Let $R_1$ and $R_2$ be connected domains intersecting $H$, $W_1$, and $H$, $W_2$, respectively and not intersecting $K_1 + K_2 + W_3 + W_4$. There is a composant $L$ of $H$ which intersects both $R_1$ and $R_2$ and lies in $M - K$. Then $L$ intersects $K_1 + K_2 + W_3 + W_4$. This is a contradiction since $M - K$ does not intersect $K_1 + K_2 + W_3 + W_4$.

Proof of Theorem 6. Suppose, on the contrary, that $M$ is the finished sum of three continua $M_1$, $M_2$, and $M_3$. One of these three continua intersects a composant of $H$ lying in $M - K$. Suppose that $M_1$ is such a continuum. Then it contains $H$ and intersects each of
the continua $M_1$ and $M_2$. Each of the continua $M_2$ and $M_1 + M_2$ contains a point of $K$ not belonging to the other of these two continua. Since the closure of $M - (M_1 + M_2)$ is a proper subset of the indecomposable continuum $K$, then $M - (M_1 + M_2)$ is not connected. Let $T_1$ and $T_2$ be two mutually separated sets whose sum is $M - (M_1 + M_2)$. Let $K_1$ and $K_2$ be two mutually exclusive simple discs whose interiors intersect $T_1$ and $T_2$ respectively such that $K_1$ and $K_2$ do not intersect $T_1 + M_1 + M_2$ and $T_1 + M_1 + M_2$ respectively. Since every composant of $K$ intersects both $K_1$ and $K_2$, there exist six distinct composants of $K$ each of which contains a continuum irreducible from $K_1$ to $K_2$. By Lemma 6.1, at most three of these intersect $H$, and hence three do not. Denote three which do not by $W_1$, $W_2$, and $W_3$. Let $D$ denote the complementary domain of $K_1 + K_2$. There exist two of the continua $W_1$, $W_3$, and $W_2$ such that their sum separates the other one from $H$ in $D$. Consider the case in which $W_1 + W_3$ separates $W_2$ from $H$ in $D$. Let $I$ denote the complementary domain of $K_1 + K_2 + W_1 + W_3$ which contains the connected set $W_2 - W_1 \cdot (K_1 + K_2)$. Since one of the sets $K_1 \cdot W_1$ and $K_2 \cdot W_2$ belongs to $T_1$ and the other to $T_2$, then $I \cdot W_2$ contains a point of the continuum $M_1 + M_2$. Since $H$ is a subset of $M_1 + M_2$ and does not intersect $I$, then there is a continuum $Z$ belonging to $I \cdot (M_1 + M_2)$ and intersecting both $W_2$ and $W_1 + W_3$. But this is impossible since $Z$ is a proper subcontinuum of $K$ intersecting two composants of $K$. Thus the supposition that $M$ is the finished sum of three continua has led to a contradiction.

**Theorem 7.** If the hypothesis of Theorem 6 is satisfied, then uncountably many composants of $K$ lie in $M - H$.

This theorem follows from Theorem 6 and [3, Theorem 1].

**Remark.** Neither Theorem 6 nor Theorem 7 holds true in Euclidean three-dimensional space. Let $H'$ be the point set obtained by translating the point set $H$ of [2, Example 1] one-half unit to the left. Let $H''$ be a point set obtained by revolving $H'$ through 90 degrees about the vertical line whose equation is $x = 1/2$. Only one composant of $H''$ intersects $H$, but every composant of $H$ intersects $H''$. It follows from [3, Theorem 1] and Theorem 2 that the continuum $H + H''$ is not indecomposable under index two.

**Added in proof.** I have recently observed that Theorem 6 follows from Theorem 1 and a lemma proved by N. E. Rutt [Some theorems on triodic continua, Amer. J. Math. vol. 56 (1934) pp. 122-132, Lemma 1]. I regret that I was not aware of Rutt's lemma at the time I prepared this paper.
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