Let $X$ be the set of all complex sequences $\alpha = \{a_n\}$ such that $||\alpha|| = \sup_n |a_n|^{1/(n+1)} < \infty$. Under the usual operations, $X$ is a complex vector space, and $||\alpha + \beta|| \leq ||\alpha|| + ||\beta||$. However, $||c\alpha||$ is seldom $|c||\alpha||$ and even though $c_n \to 0$, it is not in general true that $||c_n\alpha|| \to 0$; for example, if $\alpha = \langle 1, 1, 1, \cdots \rangle$, then $||c\alpha||$ is $c$ if $c \geq 1$ and is $1$ if $0 < c < 1$. Defining the distance between $\alpha$ and $\beta$ as $||\alpha - \beta||$, $X$ becomes a complete metric abelian group, but not a topological linear space. If with each $\alpha$ in $X$ is associated the analytic function defined at the origin by $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then this topology is that in which a sequence $\{f_n\}$ converges to the zero function only if on every bounded domain $D$, and for sufficiently large $n$, the functions $f_n$ are all analytic on $D$ and converge uniformly to zero; $f_n$ converges to $g$ if $f_n - g$ converges to zero. This topology is closely related to that introduced by Ganapathy Iyer into the vector space of all entire functions [4].

Given any $\alpha \in X$, there may be found an entire function $f(z)$ of order $1$, finite type, and such that $f(iy) = O(\exp c|y|)$, which interpolates to $\alpha$ in the sense that $f(n) = a_n$ for $n = 0, 1, 2, \cdots$ [1]. This is not the case if the condition $c < \pi$ is imposed. We have called a sequence $\alpha$ admissible in case such a more restricted function exists [2]. By a theorem of Carlson, such a function when it exists is unique [3]. A sequence $\alpha$ may fail to be admissible in an inessential way; for example, $\alpha = \langle 0, 0, 0, \cdots \rangle$ is admissible, but $\beta = \langle 1, 0, 0, \cdots \rangle$ is not. To allow for this, we shall now say that $\alpha$
is essentially admissible if there is a function \( f(z) \) with \( c < \pi \) such that 
\[ f(n) = \alpha_n \]
for all sufficiently large values of \( n \). The set of all such \( \alpha \) forms a subset of \( X \) which we denote by \( A \). Because of the theorem of Carlson cited above, one expects \( A \) to be a "sparse" subset of \( X \). The object of this note is to give this conjecture a precise form. We first observe that if \( \alpha \) and \( \beta \) lie in \( A \), so does \( \alpha + \beta \), and \( c \alpha \) for any complex \( c \). Moreover, if \( \alpha' \) is obtained from \( \alpha \) by altering any finite set of its coordinates, \( \alpha' \) is also in \( A \). The study of \( A \) may be confined to that portion of it lying in the open unit sphere, since if \( \| \alpha \| = R \) and \( \alpha \in A \), the sequence \( b_n = (1/2R)^{n+1} \alpha_n \) is also in \( A \), and \( \| \beta \| = 1/2 \).

**Theorem 1.** \( A \) is of first category in \( X \).

In spite of this, it might be supposed that \( A \) is dense in \( X \). This is not the case.

**Theorem 2.** \( A \) is not dense in \( X \). In fact, there are open subsets of the unit sphere which contain no points of \( A \).

It will be seen that there are therefore spheres of arbitrarily large size (radius) which are free of points of \( A \). We conjecture that \( A \) itself is nondense.

Before proving these, we introduce certain definitions. For any \( a \) and \( c \), let \( K(a, c) \) be the set of those entire functions \( f(z) \) obeying 
\[ f(z) = O(1) \left( \exp a|z| + c|y| \right), \]
and let \( A(a, c) \) be the corresponding subset of \( A \), comprising those sequences \( \alpha \) interpolated to by functions in \( K(a, c) \). We have \( A = \bigcup A(a, c) \) if the union is taken for all \( 0 \leq a < \alpha \) and \( 0 \leq c < \pi \). Let \( A^* \) be the union of the closures of the sets \( A(a, c) \). This of course may be only part of the closure of \( A \).

Given any \( \alpha \in X \), let \( b_n = \Delta^n a_0 = (-1)^n \sum C_{n,k} (-1)^k a_k \) and 
\[ g(z) = \sum b_n z^n. \]
We shall make use of a simple identity connecting \( g(z) \) and the function \( F(z) = \sum a_n z^n \). (See for example [5].)

\[ (1 + z)g(z) = F(z/(1 + z)), \quad (1 - z)F(z) = g(z/(1 - z)). \]

The basis for our proofs for the theorems stated above is the following characterization of the set \( A^* \).

**Theorem 3.** \( \alpha \in A^* \) if and only if \( g(z) \) is regular at the origin and has an extension to a neighborhood of \( -1 \leq x \leq 0 \) except possibly for an isolated singularity at \( -1 \).

This depends upon the following characterization for the smaller class of admissible sequences [2]: \( \alpha \) is admissible if and only if \( g(z) \) is regular on a neighborhood of the interval \( -1 \leq x \leq 0 \). This neighborhood contains an open set \( \Omega(a, c) \) depending only on the growth
constants $a$, $c$, of the entire function $f(z)$ which interpolates to $\alpha$.
Let $\alpha$ lie in the set $A(a, c)$. Choose an admissible $\beta = \{b_n\}$ such that $c_n = a_n - b_n = 0$ for all $n > n_0$. We have

$$g(z) = \sum \Delta^n b_n z^n + \sum \Delta^n c_n z^n = g_1(z) + g_2(z).$$

Since $\beta$ is admissible, $g_1$ is regular on $\Omega(a, c)$ containing $-1 \leq x \leq 0$. Since $c_n = 0$ for $n > n_0$, $F_1(z) = \sum c_n z^n$ is a polynomial, and, using (1), $g_2(z)$ has a pole at $-1$ as its only singularity. $g(z)$ is then regular in $\Omega(a, c)$ except for a pole at $-1$. Conversely, if this is true, $g(z)$ may be written as $g_1(z) + P(z)/(1+z)^m$ where $P$ is a polynomial, and $a_n = b_n + c_n$ where $\beta = \{b_n\}$ is admissible, and $c_n = 0$ for all large $n$. Let us now suppose that $\alpha$ is a limit point of $A(a, c)$. Given $\epsilon > 0$, choose $\beta \in A(a, c)$ with $||\alpha - \beta|| < \epsilon < 1$, and let $\{c_n\} = \alpha - \beta$. Then $g(z) = \sum \Delta^n b_n z^n + \sum \Delta^n c_n z^n = g_1(z) + g_2(z)$. $g_1(z)$ is regular in $\Omega(a, c)$ except for a possible pole at $-1$. Since $|c_n| < \epsilon^{n+1}$ for all $n$, $\sum c_n z^n$ is regular for $|z| < 1/\epsilon$ and, by (1), $g_2(z)$ is regular outside the disc $|z/(1+z)| \geq 1/\epsilon$. $g(z)$ is then regular on the set obtained by deleting this disc from $\Omega(a, c)$. Letting $\epsilon$ decrease, $g(z)$ is regular on all of $\Omega(a, c)$ except possibly at $-1$. Conversely, let $g(z)$ be regular on a neighborhood $\Omega$ of $[-1, 0]$ except for an isolated singularity at $-1$. Write $g(z) = g_1(z) + g_2(z) = \sum \Delta^n b_n z^n + \sum \Delta^n c_n z^n$ where $g_1$ is regular in $\Omega$, and $g_2$ has its only singularity at $-1$. Putting $\beta = \{b_n\}$, we see by the result cited above that $\beta$ is admissible, and hence in $A(a, c)$ for a suitable choice of $a$ and $c$. By (1), $\sum c_n z^n$ is entire, and $\lim |c_n|^{1/n} = 0$. Setting $\gamma = \{c_n\}$, we have $\alpha = \beta + \gamma$, with $\beta \in A(a, c)$. It is not necessarily true that $||\gamma||$ is small. However, by a slight shift, we can show that $\alpha$ lies in the closure of $A(a, c)$. For any $N$, set

$$c'_n = \begin{cases} 0, & n \leq N, \\ c_n, & n > N, \end{cases} \quad \text{and} \quad b'_n = \begin{cases} b_n + c_n, & n \leq N, \\ b_n, & n > N. \end{cases}$$

Then, $\alpha = \beta' + \gamma'$, and $||\alpha - \beta'|| = ||\gamma'|| = \sup_{n \geq N} |c_n|^{1/(n+1)}$, which may be made arbitrarily small by increasing $N$. Since $\beta'$ agrees with $\beta$ at all but a finite number of coordinates, $\beta'$ lies in $A(a, c)$.

We next proceed to the proof of Theorem 1 and Theorem 2.

**Proof of Theorem 1.** Since $A = \bigcup_{n, m} A(n, \pi - 1/m)$, we shall show that $A$ is of first category if we show that each $A(a, c)$ is nondense. Given $\beta \in A(a, c)$ and $\epsilon > 0$, we shall produce $\alpha$ such that $||\alpha - \beta|| < \epsilon$, but such that $\alpha$ is not in the closure of $A(a, c)$. Let $\gamma = \{c_n\}$ where $c_n$ is 0 if $n$ is a square, and $e^{n+1}$ otherwise, and set $\alpha = \beta + \gamma$. Clearly, $||\gamma|| = \epsilon$. The function $\sum c_n z^n = \sum e^{n+1} z^n$ has the circle $|z| = 1/\epsilon$ as a cut. By (1), $g_2(z) = \sum \Delta^n c_n z^n$ is regular for $\epsilon |z| < 1 + 1$, and the boundary of this is a cut. Since $\beta$ is in $A(a, c)$, $g_1(z) = \sum \Delta^n b_n z^n$ is
regular on a neighborhood of $-1 \leq x \leq 0$, and $g(z) = g_1(z) + g_2(z)$ does not have the type of behavior which permits $\alpha$ to be in the set $A^*$. 

**Proof of Theorem 2.** We must show that there are open sets in $X$ which contain no points of $A$. For this we use a special theorem concerning oscillating sequences: if $C_1$ and $C_2$ are two disjoint convex sets and if the terms of the sequence $\alpha$ alternate between these sets, $\alpha$ is not admissible [1]. Let $c > 1$ and consider the special sequence $\beta = \{b_n\}$ where $b_n = c(-1)^n$. If, now, $||\alpha - \beta|| \leq 1$, then $|a_n - c(-1)^n| \leq 1$ for all $n$, so that $a_n$ alternates between the circles of unit radius with center at $c$ and $-c$. Thus, the sphere, center $\beta$ and radius $1$, is disjoint from $A$. More generally, if $b_n = (-1)^n R^{n+1}$ then the open sphere, center $\beta$ and radius $R$, is disjoint from $A$. It should be noted that $||\beta|| = R$, so that these $A$-free spheres can be found at any distance from the origin.

**References**


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