ON THE SUMMABILITY OF CERTAIN SERIES FOR UNBOUNDED NONLINEAR FUNCTIONALS

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1. **Introduction.** In a previous paper[1] the authors investigated the infinite-dimensional Abel summability of the orthogonal development of a bounded nonlinear functional in terms of a set of closed orthonormal functionals.[2] In this paper we want to remove the condition of boundedness on the functional. Our main theorem is as follows:

THEOREM. *Let C be the space of all continuous functions $x(t)$ on $0 \leqq t \leqq 1$ such that $x(0)=0$ and let $L_2(C)$ be the space of all Wiener-measurable functionals of summable square. Let $F(x)$ be a functional on $L_2(C)$ such that*

$$|F(x)| \leqq B \exp \left\{ A \int_0^1 [x(t)]^2 dt \right\}, \tag{1}$$

and which is continuous in the Hilbert topology at x_0. Let

$$\Psi_{m_1,\cdots,m_N}(x) = \prod_{k=1}^{N} H_{m_k}\left(\int_0^1 2^{1/2} \cos\left(\frac{2k-1}{2}\right) \pi t dx(t) \right),$$

where H_n is the partially normalized Hermite polynomial

$$H_n(u) = (-1)^n 2^{-n/2} (n!)^{-1/2} e^{u^2} \frac{d^n}{dx^n} e^{-u^2}, \qquad n = 0, 1, 2, \cdots,$$

and let

$$A_{m_1,\cdots,m_N} = \int_C^w F(x) \Psi_{m_1,\cdots,m_N}(x) d_w x.$$

Then[3]

$$F(x_0) = \lim_{N\to\infty, \lambda \to 1-0} \sum_{m_1,\cdots,m_N=0}^{\infty} A_{m_1,\cdots,m_N} \lambda^{m_1+\cdots+m_N} \Psi_{m_1,\cdots,m_N}(x_0).$$

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[1] *On the summability of certain orthogonal developments of nonlinear functionals*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 130–145.

[2] A simpler proof of this theorem has since been given by G. Maruyama, Kōdai Mathematical Seminar Reports, no. 3, 1950, pp. 41–44.

[3] Here the double limit is to be interpreted in the sense that N never actually assumes the value $+\infty$ and λ never actually assumes the value 1.

The main theorem of the paper cited in footnote 1 is the special case of this theorem for $A=0$ and much of the argument given in that paper remains the same. In particular, it was shown that the difference $D(x_0)$ between the functional $F(x)$ and its Fourier-Hermite partial sum is given by

$$D(x_0) = c_\lambda \int_C^w [F(x) - F(x_0)] \exp\left\{ \sum_{i=1}^N \frac{2\lambda u_i\xi_i - \lambda^2(u_i^2 + \xi_i^2)}{1-\lambda^2} \right\} d_w x,$$

where

$$c_\lambda = (1-\lambda^2)^{-N/2}; \qquad u_i = \int_0^1 \alpha_i(t)dx(t);$$

$$\xi_i = \int_0^1 \alpha_i(t)dx_0(t); \text{ and } \alpha_i(t) = 2^{1/2}\cos\left(\frac{2i-1}{2}\right)\pi t, \ i = 1, 2, \cdots.$$

We sketch here the method which was used to show domination of $D(x_0)$ by a sum of integrals $\sum_{j=0}^\infty I_j$.

From continuity of $F(x)$ at $x_0 \in C$ in the Hilbert topology, it follows that if $\epsilon > 0$, there exists a $\delta > 0$ such that

$$(2) \qquad |F(x) - F(x_0)| < \frac{\epsilon}{3} \quad \text{when} \quad \int_0^1 [x(t) - x_0(t)]^2 dt \leqq \delta^2.$$

We denote by S_0 the set of $x(t)$ satisfying this last inequality:

$$S_0 = \underset{x(t)}{E}\left\{ \int_0^1 [x(t) - x_0(t)]^2 dt \leqq \delta^2 \right\},$$

and set

$$\mu = \frac{\delta}{2}\left(1 + \int_0^1 x_0^2(t)dt\right)^{-1/2};$$

then choose N_0 so large that

$$(3) \qquad \sum_{j=N_0+1}^\infty \left(\int_0^1 \beta_j(t)x_0(t)dt \right)^2 < \frac{\delta^2}{16},$$

where

$$\beta_j(t) = 2^{1/2}\sin\left(\frac{2j-1}{2}\right)\pi t, \qquad i = 1, 2, \cdots.$$

Now take

$$x_1(t) = \sum_{j=1}^{N_0} \beta_j(t) \int_0^1 \beta_j(s) x_0(s) ds;$$
$$x_2(t) = x_0(t) - x_1(t);$$

and

$$\xi_j' = \int_0^1 \alpha_j(t) dx_2(t);$$

with

$$S_j = \underset{x(t)}{E} \left\{ \left| \int_0^1 \alpha_j(t) dx(t) - \xi_j \right| > \frac{\delta}{3}\left(j - \frac{1}{2}\right)^{1/4} + \frac{\mu}{2} | \xi_j | + | \xi_j' | \right\}, \qquad i = 1, 2, \cdots;$$

and

$$(4) \quad I_j = c_\lambda \int_{S_j}^{w} | F(x) - F(x_0) | \exp\left\{ \sum_{i=1}^{N} \frac{2\lambda u_i \xi_i - \overset{2}{\lambda} (\overset{2}{u_i} + \overset{2}{\xi_i})}{1 - \lambda^2} \right\} d_w x;$$

then it was shown that

$$\left| D(x_0) \right| \leqq \sum_{j=0}^{\infty} I_j.$$

It follows, then, from (2) and (4) that $I_0 < \epsilon/3$, since the boundedness of $F(x)$ did not enter into this argument.

2. General estimation of I_j. We now continue the proof of the main theorem of this paper with a consideration of I_j for $j \neq 0$, and note that from here on the fact that $A \neq 0$ makes a difference in the details of the proof.

We shall (without loss of generality) assume that A and B of (1) are so chosen that

$$A > 1/8,$$
$$B \geqq | F(x_0) |.$$

Then by the Parseval expansion of $x(t)$,

$$| F(x) - F(x_0) | \leqq 2B \exp \left\{ A \sum_{j=1}^{\infty} \left[\int_0^1 \beta_j(t) x(t) dt \right]^2 \right\}.$$

It follows from this and Weiner's formula

$$\int_C^w f\left\{\int_0^1 \alpha_1(t)dx(t), \cdots, \int_0^1 \alpha_n(t)dx(t)\right\} d_w x$$
$$= \pi^{-n/2}\int_{-\infty}^{\infty} \overset{(n)}{\cdots} \int_{-\infty}^{\infty} f(u_1, \cdots, u_n) \exp\left\{-\sum_{k=1}^{n} u_k^2\right\} du_1 \cdots du_n,$$

that the I_j for the unbounded functional satisfy the following inequality, with $A' = A/\pi^2$:

$$\begin{aligned}
I_j &\leq 2c_\lambda B \int_{S_j}^{w} \exp\left\{\sum_{i=1}^{N} \frac{2\lambda u_i\xi_i - \lambda^2(u_i^2 + \xi_i^2)}{1-\lambda^2}\right. \\
&\quad \left. + A'\sum_{i=1}^{\infty} \frac{u_i^2}{(i-1/2)^2}\right\} d_w x \\
&= \lim_{r\to\infty} 2c_\lambda B \int_{S_j}^{w} \exp\left\{\sum_{i=1}^{N}\left[\frac{2\lambda u_i\xi_i - \lambda^2(u_i^2+\xi_i^2)}{1-\lambda^2} + \frac{A'u_i^2}{(i-1/2)^2}\right]\right. \\
&\quad \left. + A'\sum_{i=N+1}^{r} \frac{u_i^2}{(i-1/2)^2}\right\} d_w x \\
&= 2c_\lambda B \lim_{r\to\infty} \pi^{-r/2} \int_{-\infty}^{\infty} \overset{(r)}{\cdots} \int_{-\infty}^{\infty} \chi_j(v_j) \\
&\quad \cdot \prod_{i=1}^{N} \exp\left\{\frac{2\lambda u_i\xi_i - \lambda^2(v_i^2+\xi_i^2)}{1-\lambda^2} + \frac{A'v_i^2}{(i-1/2)^2} - v_i^2\right\} \\
&\quad \cdot \prod_{i=N+1}^{r} \exp\left\{\frac{A'v_i^2}{(i-1/2)^2} - v_i^2\right\} dv_1 \cdots dv_r,
\end{aligned} \tag{5}$$

where

$$\chi_j(v) = \begin{cases} 1, & \text{if } |v - \xi_j| > \dfrac{\delta}{3}\left(j - \dfrac{1}{2}\right)^{1/4} + \dfrac{\mu}{2}|\xi_j| + |\xi_j'|; \\ 0, & \text{otherwise.}\end{cases}$$

We note, for future reference, that

$$\xi_j' = \begin{Bmatrix} 0, & \text{if } j \leq N_0 \\ \xi_j, & \text{if } j > N_0 \end{Bmatrix},$$

where N_0 is given by (3). Take $N > N_0$. As in the paper cited in footnote 1 we have two cases to consider, depending on j.

3. **Estimation of I_j for the case $j \leq N$.** Here we refine our estimate

for I_j for $j \leqq N$. Then

$$
\begin{aligned}
I_j \leqq \frac{2c_\lambda B}{\pi^{1/2}} \int_{-\infty}^{\infty} \chi_j(v_j) \exp \Bigg\{ & \frac{2\lambda v_j \xi_j - \lambda^2 (v_j^2 + \xi_j^2)}{1-\lambda^2} \\
& + \frac{A' v_j^2}{(j-1/2)^2} - v_j^2 \Bigg\} dv_j
\end{aligned}
$$

$$
\begin{aligned}
(6) \qquad \cdot \prod_{i=1, i \neq j}^{N} \Bigg[\frac{1}{\pi^{1/2}} \int_{-\infty}^{\infty} \exp \Bigg\{ & \frac{2\lambda v_i \xi_i - \lambda^2 (v_i^2 + \xi_i^2)}{1-\lambda^2} \\
& + \frac{A' v_i^2}{(i-1/2)^2} - v_i^2 \Bigg\} dv_i \\
\cdot \lim_{r \to \infty} \prod_{i=N+1}^{r} \frac{1}{\pi^{1/2}} \int_{-\infty}^{\infty} \exp & \left\{ \frac{A' v_i^2}{(i-1/2)^2} - v_i^2 \right\} dv_i.
\end{aligned}
$$

This last limit becomes, if

$$
(7) \qquad N > ((A')^{1/2} + 1),
$$

$$
(8) \qquad \lim_{r \to \infty} \prod_{i=N+1}^{r} \frac{1}{\left(1 - \dfrac{A'}{(i-1/2)^2}\right)^{1/2}} = \left[\prod_{i=N+1}^{\infty} \left(1 - \frac{A'}{(i-1/2)^2}\right) \right]^{-1/2}.
$$

We shall need certain estimates related to the error function and we consider this next. Now

$$
\begin{aligned}
\int_{\theta}^{\infty} e^{-\alpha v^2 + \beta v + \gamma} dv &= e^{\gamma + \beta^2/4\alpha} \int_{\theta}^{\infty} e^{-\alpha(v - \beta/2\alpha)^2} dv \\
&= e^{\gamma + \beta^2/4\alpha} \frac{1}{\alpha^{1/2}} \operatorname{erfc}(\Theta),
\end{aligned}
$$

where

$$
(9) \qquad \Theta = \theta \alpha^{1/2} - \frac{\beta}{2\alpha^{1/2}}.
$$

Hence, if we take

$$
\alpha = \frac{1}{1-\lambda^2} - \frac{A'}{(i-1/2)^2}, \qquad \beta = \frac{2\lambda\xi}{1-\lambda^2}, \qquad \gamma = \frac{-\lambda^2\xi^2}{1-\lambda^2},
$$

we have for one of the integrals to be evaluated in (6)

$$
\begin{aligned}
(10)\qquad & \frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty}\exp\left\{\frac{2\lambda v\xi-\lambda^2(v^2+\xi^2)}{1-\lambda^2}+\frac{A'v^2}{(i-1/2)^2}-v^2\right\}dv \\
& = \exp\left\{\frac{A'\lambda^2\xi^2}{(i-1/2)^2-A'(1-\lambda^2)}\right\}\left(\frac{(1-\lambda^2)(i-1/2)^2}{(i-1/2)^2-A'(1-\lambda^2)}\right)^{1/2},
\end{aligned}
$$

and for another

$$
\begin{aligned}
& \frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty}\chi_j(v)\exp\left\{\frac{2\lambda v\xi_j-\lambda^2(v^2+\xi_j^2)}{1-\lambda^2}-\frac{A'v^2}{(j-1/2)^2}-v^2\right\}dv \\
(11)\qquad & = \exp\left\{\frac{A'\lambda^2\xi_j^2}{(i-1/2)^2-A'(1-\lambda^2)}\right\}\left(\frac{(1-\lambda^2)(j-1/2)^2}{(j-1/2)^2-A(1-\lambda^2)}\right)^{1/2} \\
& \cdot\frac{1}{\pi^{1/2}}\left[\operatorname{erfc}(\Theta_j')+\operatorname{erfc}(-\Theta_j'')\right],
\end{aligned}
$$

where Θ_j' and Θ_j'' are related to θ_j' and θ_j'' respectively as in (9), and

$$
\begin{aligned}
(12)\qquad \theta_j' &= \xi_j+\frac{\delta}{3}\left(j-\frac{1}{2}\right)^{1/4}+\frac{1}{2}\mu|\xi_j|+|\xi_j'|, \\
\theta_j'' &= \xi_j-\frac{\delta}{3}\left(j-\frac{1}{2}\right)^{1/4}-\frac{1}{2}\mu|\xi_j|-|\xi_j'|.
\end{aligned}
$$

Putting (10), (11), and (8) in (6), we obtain, when $A'(1-\lambda^2)<1/4$ and (7) holds,

$$
\begin{aligned}
& I_j \leqq 2c_\lambda B(1-\lambda^2)^{N/2}\prod_{i=1}^{N}\left[1-\frac{A'(1-\lambda^2)}{(j-1/2)^2}\right]^{-1/2} \\
(13)\qquad & \cdot\exp\left\{A'\lambda^2\sum_{i=1}^{\infty}\frac{\xi_i^2}{(i-1/2)^2-A'(1-\lambda^2)}\right\} \\
& \cdot\frac{1}{\pi^{1/2}}\left\{\operatorname{erfc}(-\Theta_j'')+\operatorname{erfc}(\Theta_j')\right\} \\
& \cdot\prod_{i=N+1}^{\infty}\left(1-\frac{A'}{(i-1/2)^2}\right)^{-1/2}.
\end{aligned}
$$

Then if

$$
1>\lambda>(1-1/8A')^{1/2},
$$

we have

$$A'(1-\lambda^2) < \frac{1}{8} \leqq \frac{1}{2}\left(i - \frac{1}{2}\right)^2, \qquad i = 1, 2, \cdots . \tag{14}$$

Hence

$$\begin{aligned}\sum_{i=1}^{\infty} \frac{\xi_i^2}{(i-1/2)^2 - A'(1-\lambda^2)} &\leqq \sum_{i=1}^{\infty} \frac{\xi_i^2}{(i-1/2)^2}\cdot\frac{(i-1/2)^2}{(i-1/2)^2 - A'(1-\lambda^2)} \\ &\leqq 2\sum_{i=1}^{\infty}\frac{\xi_i^2}{(i-1/2)^2} \\ &= 2\sum_{i=1}^{\infty}\frac{\left[\int_0^1 \alpha_i(t)dx_0(t)\right]^2}{(i-1/2)^2} \\ &= 2\pi^2\sum_{i=1}^{\infty}\left[\int_0^1 \beta_i(t)x_0(t)dt\right]^2 \\ &= 2\pi^2\int_0^1 [x_0(t)]^2 dt.\end{aligned}$$

Thus

$$\begin{aligned}&\exp\left\{A'\lambda^2\sum_{i=1}^{\infty}\frac{\xi_i^2}{(i-1/2)^2 - A'(1-\lambda^2)}\right\} \\ &\qquad\leqq \exp\left\{2A'\pi^2\int_0^1 [x_0(t)]^2 dt\right\} = B_1\end{aligned} \tag{15}$$

a constant depending only on A' and $x_0(t)$.

Assuming

$$N > (A')^{1/2} + 1,$$

we have $N+1 > (A')^{1/2}+2 \geqq [(A')^{1/2}]+2$. Hence the set of integers $i \geqq N+1$ is a subset of the set of integers $i \geqq [(A')^{1/2}]+2$. Moreover, if $i \geqq [(A')^{1/2}]+2$, we have

$$i \geqq (A')^{1/2}+1, \text{ and } i - 1/2 > (A')^{1/2},$$

so

$$\left[1 - \frac{A'}{(i-1/2)^2}\right]^{-1/2} > 1,$$

$$\prod_{i=N+1}^{\infty}\left[1-\frac{A'}{(i-1/2)^2}\right]^{-1/2} \leq \prod_{i=[(A')^{1/2}]+2}^{\infty}\left[1-\frac{A'}{(i-1/2)^2}\right]^{-1/2} = B_3 \tag{16}$$

for all $N>(A')^{1/2}+1$, where B_3 is a constant dependent only on A'.

It follows from (14) that

$$1-\frac{A'(1-\lambda^2)}{(i-1/2)^2} > 1-\frac{1}{8(i-1/2)^2},$$

so

$$\begin{aligned}\prod_{i=1}^{N}\left[1-\frac{A'(1-\lambda^2)}{(i-1/2)^2}\right]^{-1/2} &< \prod_{i=1}^{N}\left[1-\frac{1}{8(i-1/2)^2}\right]^{-1/2}\\ &< \prod_{i=1}^{\infty}\left[1-\frac{1}{8(i-1/2)^2}\right]^{-1/2}\\ &= \left[\cos\left(\frac{\pi}{2^{3/2}}\right)\right]^{-1/2} = B_2\end{aligned} \tag{17}$$

where B_2 is a constant independent of N and λ. By (9)

$$\begin{aligned}\Theta_j' &= \theta_j'\alpha^{1/2} - \frac{\beta}{2\alpha^{1/2}} = \theta_j'\left(\frac{1}{1-\lambda^2}-\frac{A'}{(j-1/2)^2}\right)^{1/2}\\ &\quad - \frac{\lambda\xi_j}{(1-\lambda^2)\left(\frac{1}{1-\lambda^2}-\frac{A'}{(j-1/2)^2}\right)^{1/2}}\\ &= \frac{1}{(1-\lambda^2)^{1/2}}\left(1-\frac{A'(1-\lambda^2)}{(j-1/2)^2}\right)^{1/2}\left[\theta_j' - \frac{\lambda\xi_j}{1-\frac{A'(1-\lambda^2)}{(j-1/2)^2}}\right].\end{aligned}$$

Note that $1-A'(1-\lambda^2)/(j-1/2)^2>1/2$ by (14). Then from (12)

$$\begin{aligned}\Theta_j' = \frac{1}{(1-\lambda^2)^{1/2}}\left(1-\frac{A'(1-\lambda^2)}{(j-1/2)^2}\right)^{1/2}\Bigg[\xi_j + \frac{\delta}{3}\left(j-\frac{1}{2}\right)^{1/4}\\ + \frac{1}{2}\mu\left|\xi_j\right| + \left|\xi_j'\right| - \frac{\lambda\xi_j}{1-\frac{A'(1-\lambda^2)}{(j-1/2)^2}}\Bigg].\end{aligned}$$

Now let $A'(1-\lambda^2)/(j-1/2)^2=a$. By (14), $a<1/2$, so that

$$\left|\frac{\lambda-1+a}{1-a}\right| < 2|\lambda-1+a| = 2\left|\lambda-1+\frac{A'(1-\lambda^2)}{(j-1/2)^2}\right|$$
$$= 2(1-\lambda)\left|\frac{A'(1+\lambda)}{(j-1/2)^2}-1\right| < 2(1-\lambda)(8A'+1).$$

Now if λ is so chosen that

$$(18) \qquad 1 > \lambda > 1 - \mu/4(8A'+1),$$

it follows that

$$(19) \qquad \frac{1}{2}\mu > \left|\frac{\lambda-1+a}{1-a}\right|.$$

Hence

$$\frac{1}{2}\mu|\xi_j| \geqq \frac{-\xi_j(1-\lambda)+a\xi_j}{1-a}$$

and

$$\xi_j + \frac{1}{2}\mu|\xi_j| + |\xi_j'| - \frac{\lambda\xi_j}{1-a} \geqq 0.$$

Thus

$$\Theta_j' > \frac{1}{(1-\lambda^2)^{1/2}}\cdot\frac{\delta}{3\cdot 2^{1/2}}\left(j-\frac{1}{2}\right)^{1/4}.$$

Again, by (9),

$$\Theta_j'' = \theta_j''\alpha^{1/2} - \frac{\beta}{2\alpha^{1/2}}$$
$$(20) \qquad = \frac{(1-a)^{1/2}}{(1-\lambda^2)^{1/2}}\left[\xi_j - \frac{\delta}{3}\left(j-\frac{1}{2}\right)^{1/4} - \frac{1}{2}\mu|\xi_j| - |\xi_j'| - \frac{\lambda\xi_j}{1-a}\right].$$

Multiplying (19) by $|\xi_j|$ gives

$$\frac{1}{2}\mu|\xi_j| > \left|\frac{(\lambda-1)\xi_j+a\xi_j}{1-a}\right| \geqq \frac{(1-\lambda)\xi_j-a\xi_j}{1-a} = \xi_j - \frac{\lambda\xi_j}{1-a}.$$

$$\xi_j - \frac{1}{2}\mu|\xi_j| - |\xi_j'| - \frac{\lambda\xi_j}{1-a} \leqq 0,$$

and this with (20) gives

$$\Theta_j'' \leqq \frac{-1}{(1-\lambda^2)^{1/2}} \frac{\delta}{3\cdot 2^{1/2}}\left(j-\frac{1}{2}\right)^{1/4}.$$

Putting in (13) the estimates for each of the factors (17), (15), and (16), we have

$$\begin{aligned} I_j &\leqq 2BB_1B_2B_3\frac{2}{\pi^{1/2}}\operatorname{erfc}\left\{\frac{1}{(1-\lambda^2)^{1/2}}\frac{\delta}{3\cdot 2^{1/2}}\left(j-\frac{1}{2}\right)^{1/4}\right\} \\ &= B_4\int_{\delta(j-1/2)^{1/4}/3\cdot 2^{1/2}(1-\lambda^2)^{1/2}}^{+\infty} e^{-x}\,dx, \end{aligned} \tag{21}$$

our final estimate for the first case.

4. **Estimation of I_j for the case $j>N$.** Since $N>N_0$, we also have $j>N_0$. Now by (5)

$$\begin{aligned} I_j \leqq 2c_\lambda B \prod_{i=1}^{N}\frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty}\exp\cdot\left\{\frac{2\lambda v_i\xi_i-\lambda^2(v_i^2+\xi_i^2)}{1-\lambda^2}\right. \\ \left. +\frac{A'v_i^2}{(i-1/2)^2}-v_i^2\right\}dv_i \end{aligned} \tag{22}$$

$$\cdot\frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty}\chi_j(v_j)\exp\cdot\left\{\frac{A'v_j^2}{(j-1/2)^2}-v_j^2\right\}dv_j \tag{23}$$

$$\lim_{r\to\infty}\prod_{i=N+1,i\neq j}^{r}\left[\frac{1}{\pi^{1/2}}\int_{-\infty}^{\infty}\exp\cdot\left\{\frac{A'v_i^2}{(i-1/2)^2}-v_i^2\right\}dv_i\right]. \tag{24}$$

The limit above equals, if $N>(A')^{1/2}+1$, as in (15),

$$\lim_{r\to\infty}\prod_{i=N+1,i\neq j}^{r}\left[1-\frac{A'}{(i-1/2)^2}\right]^{-1/2} = \left[\prod_{i=N+1,i\neq j}^{\infty}\left(1-\frac{A'}{(i-1/2)^2}\right)\right]^{-1/2}.$$

Now set

$$v_j\left(1-\frac{A'}{(j-1/2)^2}\right)^{1/2} = v_jR_j = s$$

so that

$$R_j^2 = 1-A'/(j-1/2)^2.$$

Then (23) becomes

$$(25)\quad \frac{1}{R_j\pi^{1/2}}\int_{-\infty}^{\infty}\chi_j\left(\frac{s}{R_j}\right)e^{-s^2}ds = \frac{1}{R_j\pi^{1/2}}\left(\int_{\theta_j' R_j}^{\infty} + \int_{-\theta_j'' R_j}^{\infty}\right)e^{-s^2}ds.$$

But

$$\begin{aligned}\theta_j' &= \xi_j + \frac{\delta}{3}\left(j - \frac{1}{2}\right)^{1/4} + \frac{1}{2}\mu|\xi_j| + |\xi_j'| \\ &= \xi_j + \frac{\delta}{3}\left(j - \frac{1}{2}\right)^{1/4} + \frac{1}{2}\mu|\xi_j| + |\xi_j| \\ &\geqq \frac{\delta}{3}\left(j - \frac{1}{2}\right)^{1/4},\end{aligned}$$

and similarly,

$$\theta_j'' \leqq -(\delta/3)(j-1/2)^{1/4}.$$

Then for $j>N>(2A')^{1/2}+1$, $A'/(j-1/2)^2<1/2$, and $R_j>1/2^{1/2}$. So the first member of (25) is

$$\leqq \frac{2}{R_j\pi^{1/2}}\int_{(\delta/3\cdot 2^{1/2})(j-1/2)^{1/4}}^{\infty} e^{-s^2}ds.$$

The product of (23) and (24), then, is

$$\begin{aligned}&\leqq \frac{2}{\pi^{1/2}}\left(\prod_{i=N+1}^{\infty} R_i\right)^{-1}\operatorname{erfc}\left\{\frac{\delta}{3\cdot 2^{1/2}}\left(j-\frac{1}{2}\right)^{1/4}\right\} \\ (26)\quad &\leqq \frac{2}{\pi^{1/2}}B_3\operatorname{erfc}\left\{\frac{\delta}{3\cdot 2^{1/2}}\left(j-\frac{1}{2}\right)^{1/4}\right\},\end{aligned}$$

since (16) still holds. Now each of the N integrals in (22) is of the type already evaluated in (10), so by this and (26) we have

$$\begin{aligned}I_j \leqq\ & 2c_\lambda B\exp\cdot\left\{A'\lambda^2\sum_{i=1}^{N}\frac{\xi_i^2}{(i-1/2)^2 - A'(1-\lambda^2)}\right\} \\ &\cdot\prod_{i=1}^{N}\left(\frac{(1-\lambda^2)(i-1/2)^2}{(i-1/2)^2 - A'(1-\lambda^2)}\right)^{1/2} \\ &\cdot\frac{2}{\pi^{1/2}}B_3\operatorname{erfc}\left\{\frac{\delta}{3\cdot 2^{1/2}}\left(j-\frac{1}{2}\right)^{1/4}\right\} \\ \leqq\ & \frac{4}{\pi}BB_3B_1B_2\operatorname{erfc}\left\{\frac{\delta}{3\cdot 2^{1/2}}\left(j-\frac{1}{2}\right)^{1/4}\right\},\end{aligned}$$

by (15) and (17). Thus

$$
(27)\qquad \begin{aligned} I_j &\leqq B_4 \operatorname{erfc}\left\{\frac{\delta}{3\cdot 2^{1/2}}\left(j-\frac{1}{2}\right)^{1/4}\right\} \\ &= B_4\int_{(\delta/3\cdot 2^{1/2})(j-1/2)^{1/4}}^{\infty} e^{-v^2}dv. \end{aligned}
$$

This is our final estimate for the second case.

5. **Final estimate for** $D(x_0)$. We combine the estimates (21) and (27) and assume for this that $N>\max\{N_0, (2A')^{1/2}+1\}$, and, further, that

$$
1>\lambda>\max\left\{\left(1-\frac{1}{8A'}\right)^{1/2}, 1-\frac{\mu}{4(8A'+1)}\right\}.
$$

Thus (14) and (18) hold and hence the estimates (21) and (27). Thus

$$
\sum_{j=1}^{\infty} I_j \leqq B_4\left[\sum_{j=1}^{N}\operatorname{erfc}\left\{\frac{\delta(j-1/2)^{1/4}}{3\cdot 2^{1/2}(1-\lambda^2)^{1/2}}\right\} + \sum_{j=N+1}^{\infty}\operatorname{erfc}\left\{\frac{\delta(j-1/2)^{1/4}}{3\cdot 2^{1/2}}\right\}\right].
$$

Since

$$
\operatorname{erfc}(M)=\int_M^{\infty} e^{-v^2}dv<\frac{e^{-M^2}}{2M}\qquad (M>0),
$$

$$
(28)\qquad \begin{aligned} \sum_{j=1}^{\infty} I_j < \frac{B_4}{2}\sum_{j=1}^{\infty}\frac{3\cdot 2^{1/2}(1-\lambda^2)^{1/2}}{\delta(j-1/2)^{1/4}}\exp\left\{\frac{-\delta^2(j-1/2)^{1/2}}{18(1-\lambda^2)}\right\} \\ +\frac{B_4}{2}\sum_{j=N+1}^{\infty}\frac{3\cdot 2^{1/2}}{\delta(j-1/2)^{1/4}}\exp\left\{\frac{-\delta^2(j-1/2)^{1/2}}{18}\right\}. \end{aligned}
$$

Since the sums on the right converge, we may take N_ϵ large enough to make the last term less than $\epsilon/3$ for $N>N_\epsilon$.

Moreover the first series on the right converges and decreases term by term as $\lambda\to 1-0$, and has the limit 0. We may therefore choose λ_ϵ in the interval

$$
1>\lambda_\epsilon>\max\left\{\left(1-\frac{1}{8A'}\right)^{1/2}, 1-\frac{\mu}{4(8A'+1)}\right\},
$$

so that the first term on the right is less than $\epsilon/3$ when $\lambda_\epsilon<\lambda<1$. Thus when

$$N > N_\epsilon \quad \text{and} \quad \lambda_\epsilon < \lambda < 1,$$

$$|D(x_0)| \leqq \sum_{j=0}^{\infty} I_j < \epsilon,$$

by the above inequalities and (4.1) of the paper cited in footnote 1. The same corollary as in that paper holds now for the functional of (1)—and for the same reasons as given in the proof for the bounded functional.

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ESSENTIALLY ADMISSIBLE SEQUENCES

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Let X be the set of all complex sequences $\alpha = \{a_n\}$ such that $\|\alpha\| = \sup_n |a_n|^{1/(n+1)} < \infty$. Under the usual operations, X is a complex vector space, and $\|\alpha+\beta\| \leqq \|\alpha\| + \|\beta\|$. However, $\|c\alpha\|$ is seldom $|c|\,\|\alpha\|$ and even though $c_n \to 0$, it is not in general true that $\|c_n\alpha\| \to 0$; for example, if $\alpha = \langle 1, 1, 1, \cdots \rangle$, then $\|c\alpha\|$ is c if $c \geqq 1$ and is 1 if $0 < c < 1$. Defining the distance between α and β as $\|\alpha - \beta\|$, X becomes a complete metric abelian group, but not a topological linear space. If with each α in X is associated the analytic function defined at the origin by $f(z) = \sum a_n z^n$, then this topology is that in which a sequence $\{f_n\}$ converges to the zero function only if on every bounded domain D, and for sufficiently large n, the functions f_n are all analytic on D and converge uniformly to zero; f_n converges to g if $f_n - g$ converges to zero. This topology is closely related to that introduced by Ganapathy Iyer into the vector space of all entire functions [4].

Given any $\alpha \in X$, there may be found an entire function $f(z)$ of order 1, finite type, and such that $f(iy) = O(\exp c|y|)$ for some $c \leqq \pi$, which interpolates to α in the sense that $f(n) = a_n$ for $n = 0, 1, 2, \cdots$ [1]. This is not the case if the condition $c < \pi$ is imposed. We have called a sequence α admissible in case such a more restricted function exists [2]. By a theorem of Carlson, such a function when it exists is unique [3]. A sequence α may fail to be admissible in an inessential way; for example, $\alpha = \langle 0, 0, 0, \cdots \rangle$ is admissible, but $\beta = \langle 1, 0, 0, \cdots \rangle$ is not. To allow for this, we shall now say that α

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