ON UNIFORM DISTRIBUTION OF ALGEBRAIC NUMBERS

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Let \( K \) be an algebraic number field of degree \( n+1 \) over the rational field \( \mathbb{R} \). The conjugates of \( K \) are denoted by \( K^{(0)}, \ldots, K^{(n)} \), it being supposed that \( K^{(0)}, \ldots, K^{(r_1)} \) are real, \( K^{(r_1+1)}, \ldots, K^{(n)} \) are complex, and moreover that for \( k = r_1 + 1, \ldots, r_1 + r_2 \) the fields \( K^{(k)} \) and \( K^{(k+r_3)} \) are obtainable from one another by interchanging \( i = (-1)^{1/2} \) and \( -i \) (here \( r_1 + 2r_2 = n \)). I assume throughout that \( r_1 \geq 0 \) and that \( n \geq 1 \). Numbers in \( K \) are represented by small Greek letters, the insertion of superscripts denoting the passage to the corresponding conjugates. Let \( \omega_0, \ldots, \omega_n \) be a set of numbers in \( K \) which are linearly independent over \( \mathbb{R} \); let \( \omega_0 = 1 \); and finally let \( f(x_1, \ldots, x_n) \) be a (complex-valued) function of \( n \) real variables which is Riemann integrable over the unit cube \( E_n: 0 < x_k < 1 \) \((k = 1, \ldots, n)\). Then, according to theorems of Weyl on uniform distribution,

\[
\int_{E_n} f(x_1, \ldots, x_n) \, dx_1 \cdots dx_n = \frac{1}{M} \sum_{m=0}^{M-1} f(\{m\omega_1^{(0)}\}, \ldots, \{m\omega_n^{(0)}\}) + o(1)
\]

as \( M \to \infty \), the braces denoting the fractional part. The object of this paper is to show that the error term in (1) may be replaced by \( O(1/M) \) provided \( f \) satisfies certain additional conditions.

More precisely, suppose that in \( E_n \)

\[
f(x_1, \ldots, x_n) = \sum_{q_1, \ldots, q_n = -\infty}^{\infty} a(q_1, \ldots, q_n) e^{2i\pi (q_1 x_1 + \cdots + q_n x_n)},
\]

the series on the right being absolutely convergent.

**Theorem.** If there exist positive constants \( C \) and \( c \) such that

\[
|a(q_1, \ldots, q_n)| < C (|q_1| + \cdots + |q_n|)^{-n-c},
\]

then the expression

\[
S_M = M \int_{E_n} f(x_1, \ldots, x_n) \, dx_1 \cdots dx_n
\]

\[
- \sum_{m=0}^{M-1} f(m\omega_1^{(0)}, \ldots, m\omega_n^{(0)})
\]

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is bounded in absolute value with a bound which depends only on $C$, $c$, $\omega_1$, $\cdots$, $\omega_n$ (and not on $M$).

The theorem probably remains true when (3) is replaced by the weaker condition

$$|a(q_1, \cdots, q_n)| < C(1 + |q_1|)^{-1-\epsilon} \cdots (1 + |q_n|)^{-1-\epsilon},$$

but I have been unable to prove this.

For the proof of the theorem, it is clear from (2) and (4) that

$$S_M = \sum' a(q_1, \cdots, q_n) \frac{\sin \pi M(q_1\omega_1^{(0)} + \cdots + q_n\omega_n^{(0)})}{\sin \pi(q_1\omega_1^{(0)} + \cdots + q_n\omega_n^{(0)})} e^{i\pi(M-1)(q_1\omega_1^{(0)} + \cdots + q_n\omega_n^{(0)})}$$

where the $'$ indicates that $q_1 = \cdots = q_n = 0$ is to be omitted in summation. Thus, from (3),

$$|S_M| < \frac{C}{2} \sum' (|q_1| + \cdots + |q_n|)^{-n-\epsilon}$$

where $||x||$ stands for $\min_{k=0, \pm 1, \pm 2, \ldots} |x-k|$.

Let $\mathfrak{A}$ denote the set of numbers $\alpha$ in $K$ of the form

$$\alpha = q_0 + q_1\omega_1 + \cdots + q_n\omega_n$$

with rational integers $q_0, \cdots, q_n$. An element $\alpha$ of $\mathfrak{A}$ is uniquely determined by the numbers $q_1, \cdots, q_n$ and the condition

$$|\alpha^{(0)}| < 1/2.$$ When this condition is satisfied, one sees that

$$|q_0| < \max_{k=1, \cdots, n} |\omega_k^{(0)}| \cdot (|q_1| + \cdots + |q_n|) + 1/2,$$

i.e.,

$$|q_0| < C_1(|q_1| + \cdots + |q_n|),$$

whence

$$|\alpha^{(k)}| < C_2(|q_1| + \cdots + |q_n|) \quad (k = 1, \cdots, n)$$

with

$$C_2 = 1/2 + \max_{k=1, \cdots, n} |\omega_k^{(0)}| + \max_{j,k=1, \cdots, n} |\omega_k^{(j)}|.$$
Thus,

\[ |S_M| < C_S \sum_{\alpha \in \mathbb{F}, \theta > |\alpha(\theta)| < 1/2} |N(\alpha)|^{-1} \left( \max_{k=1, \ldots, n} |\alpha^{(k)}| \right)^{-\varepsilon}. \]

The ideal \((1, \omega_1, \ldots, \omega_n)\) may be written \(a/m\) where \(a\) is an integral ideal and \(m\) is a positive rational integer. The elements of \(\mathbb{F}\) are, of course, all divisible by \(a/m\); hence

\[ (5) \quad |S_M| < C_4 \sum_{\alpha|\theta, \theta < |\alpha(\theta)| < m/2} |N(\alpha)|^{-1} \left( \max_{k=1, \ldots, n} |\alpha^{(k)}| \right)^{-\varepsilon}. \]

It is, therefore, sufficient to prove that the sum on the right side of (5), with \(a = (1)\), is convergent.

Let \(P\) be a complete set of nonassociated integers of \(K\), so that every integer of \(K\) is uniquely expressible as the product of a unit by an element of \(P\). Then the sum in (5) with \(a = (1)\) may be written

\[ S = \sum_{\alpha \in P} |N(\alpha)|^{-1} \sum_{|N(\alpha)| = 1, \theta < |\epsilon^{(k)}\alpha^{(k)}| < m/2} \left( \max_{k=1, \ldots, n} |\epsilon^{(k)}\alpha^{(k)}| \right)^{-\varepsilon}, \]

where \(\epsilon\), in the inner sum, runs over all units satisfying the indicated condition. If the elements \(\alpha\) of \(P\) are chosen so that

\[ C_6 |N(\alpha)|^{1/n} > |\alpha^{(k)}| > |N(\alpha)|^{1/n} \quad (k = 1, \ldots, n) \]

(which according to the Dirichlet theory of units is always possible for a constant \(C_6\) which depends only on the regulator of \(K\)), then

\[ (6) \quad S \leq \sum_{\alpha \in P} |N(\alpha)|^{-1} \frac{1}{n} \sum_{|N(\alpha)| = 1, |\epsilon(\alpha)| < m/2} \left( \max_{k=1, \ldots, n} |\epsilon^{(k)}\alpha^{(k)}| \right)^{-\varepsilon}, \]

where \(\xi_K\) is the Dirichlet zeta function for the field \(K\) and \(C_6 = mC_6/2\) is a constant which depends only on the numbers \(\omega_1, \ldots, \omega_n\) and on the regulator of \(K\).

Appealing once more to the Dirichlet theory of units, one sees from the \(r = r_1 + r_2\)-dimensional lattice formed by the logarithms of the units that there is a constant \(C_7 > 0\) such that at most one unit \(\epsilon\) (aside from multiplicative factors of roots of unity) satisfies

\[ (7) \quad t_k \leq \log |\epsilon^{(k)}| < t_k + C_7 \quad (k = 1, \ldots, n), \]

the real numbers \(t_1, \ldots, t_n\) being arbitrarily prescribed in advance with the convention that \(t_{k+r_2} = t_k\) for \(k = r_1 + 1, \ldots, r_1 + r_2\). Note that
is a consequence of (7).

Since \( r_1 \geq 0 \), the only roots of unity in \( K \) are \( \pm 1 \); therefore, choosing

\[
t_k = m_k C_i \quad (k = 1, \ldots, r)
\]

where \( m_1, \ldots, m_r \) are rational integers, one deduces from (6), (7), (8) that

\[
S < 2 \xi_k \left( 1 + \frac{c}{n} \right) \sum e^{-c C_i \max (m_1, \ldots, m_r)}
\]

where the sum runs over all sets of rational integers \( m_1, \ldots, m_r \) such that

\[
-(m_{r_1} + \cdots + m_{r_2}) - 2(m_{r_2+1} + \cdots + m_{r_t+1}) < \log \frac{C_8}{C_i} = -C_8,
\]

say. Thus

\[
S < 2 \frac{r_1 r_2}{r_{t+1}} \left( 1 + \frac{c}{n} \right) \sum_{m_1 \leq m_2 \leq \cdots \leq m_{r_1}, \ldots, m_{r_t+1}, \ldots, m_r} e^{-c C_i \max (m_1, m_{r_1+1})}.
\]

The latter sum may be divided into two sections in the first of which \( m_1 \geq m_{r_1+1} \) and in the second of which \( m_{r_1+1} > m_1 \).

In the first section the inequalities under the summation sign imply \( m_{r_1} > C_8 - (n - 1)m_1 \) and \( m_r > (C_8 - (n - 2)m_1)/2 \) so that the number of summands corresponding to a particular \( m_1 \) is at most \( (nm_1 - C_8)^{-1} 2^{-n} \) or 0 according as \( m_1 > C_8/n \) or \( m_1 \leq C_8/n \). In the second section of the sum, the inequalities imply \( m_{r_1} > C_8 - (n - 1)m_{r_1+1} \) and \( m_r > (C_8 - (n - 2)m_{r_1+1})/2 \), so that the number of summands corresponding to a particular \( m_{r_1+1} \) is at most \( (nm_{r_1+1} - C_8)^{-1} 2^{-n+1} \) or 0 according as \( m_{r_1+1} > C_8/n \) or \( m_{r_1+1} \leq C_8/n \). Thus,

\[
S < 3 \frac{2^{-n+1} r_1 r_2}{r_{t+1}} \left( 1 + \frac{c}{n} \right) \sum_{m=1+[C_8/n]}^{\infty} (nm - C_8)^{-1} e^{-m \cdot C_i}
\]

which is clearly finite.

It is interesting to observe, from (9), that the bound for \( S \) is \( O(c^{-1}) \).

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