REAL-VALUED FUNCTIONS ON PARTIALLY ORDERED SETS

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It is known that if P is a partially ordered set, then P can be imbedded into an everywhere branching partially ordered set Q in such a manner that if a function has a limit L on P, the function can be extended to Q and have a limit L on Q.[1] The purpose of this note is to show that P can be imbedded isomorphically into an everywhere branching partially ordered set Q and each function f on P extended to Q, in such a manner that f has a limit L on P if and only if it has a limit L on Q.[2]

By a partially ordered set is meant a set of elements $P = \{p\}$, with a binary relation "$\leqq$" which has the three properties:

(1) $p \leqq p$ for each element p of P;

(2) if $p_1 \leqq p_2$ and $p_2 \leqq p_3$, then $p_1 \leqq p_3$; and

(3) if $p_1 \leqq p_2$ and $p_2 \leqq p_1$, then $p_1 = p_2$ (identity).

As usual, "$p_1 < p_2$" will mean that $p_1 \leqq p_2$, but p_1 is not identical with p_2. An element p_0 of P is called a minimal (maximal) element of P if there is no element p of P for which $p < p_0$ ($p > p_0$). The only partially ordered sets which are considered hereafter are those which have no minimal elements. A partially ordered set is directed if, for each pair of elements in P, p_1 and p_2, an element p_3 can be found for which $p_3 \leqq p_i$, $i = 1, 2$. A partially ordered set is everywhere branching if, to each element p_1 of P, there corresponds a pair of elements p_2 and p_3 such that $p_i \leqq p_1$, $i = 2, 3$, and

$$\{p \mid p \leqq p_2\} \cap \{p \mid p \leqq p_3\} = \emptyset$$

where $\emptyset$ is the empty set. A subset $Q = \{q\}$ of P is a coinitial subset of P if to each element p in P, there corresponds an element q in Q such that $q \leqq p$. $Q = \{q\}$ is a residual subset of P if, for each element q in Q, $\{p \mid p \leqq q, p \in P\}$ is a subset of Q.

A single, real-valued function f defined on a partially ordered set $P = \{p\}$ has a limit L if, to each element p_0 of P, and $\epsilon > 0$, there corresponds an element $p_1(p_0, \epsilon)$ of P such that $p_1 \leqq p_0$ and $|f(p) - L| < \epsilon$ for $p \leqq p_1$.[3]

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[1] See Day, *Oriented systems*, Duke Math. J. vol. 11 (1944) p. 201 ff.

[2] The author wishes to thank the referee for his general suggestions, particularly in the simplification of the proof of Theorem 1.

[3] See Alaoglu and, Birkhoff, *General ergodic theorems*, Ann. of Math. vol. 41 (1940) pp. 293–309.

A single, real-valued function f defined on a partially ordered set P has a partial limit L on P if, for some residual subset Q of P, the function f, considered as a function on Q, has a limit L.

THEOREM 1. *A partially ordered set $P=\{p\}$ may be imbedded into an everywhere branching partially ordered set $Q=\{q\}$ by an isomorphism g.*[4] *Furthermore, a function h of Q onto $g(P)$ can be found which has the following two properties:*

(1) *$h[g(p)]=g(p)$ for each p in P; and*

(2) *f being any real function on P, then the function f_*, which is defined by* (a) *$f_*[g(p)]=f(p)$ for p in P, and* (b) *$f_*(q)=f_*[h(q)]$ for q in Q, has a limit L on Q if and only if f has a limit L on P.*

PROOF. For each p in P let $g(p)=\{x \mid x\leqq p,\ x\in P\}$. Let

$$\{Q = q \mid q \text{ is a coinitial subset of } g(p),\ p \in P\},$$

and order the elements of Q by set inclusion. To see that Q is an everywhere branching partially ordered set, well order the elements of P into a sequence, say $\{r_\xi\}$, $\xi<\gamma$. A simply ordered subset $A=\{a\}$ of a partially ordered set $B=\{b\}$ shall be called a path (in B) if there is no element b_0 of B such that $b_0\leqq a$ for each element a in A. Clearly, if b_0 is any element of B, then there exists a path in B,

$$a_0 > a_1 > \cdots > a_\xi > \cdots .$$

Now let $q=\{y\}$ be any element of Q. Let y_0^0 be the first element of q and

$$y_0^0 > y_1^0 > \cdots > y_\xi^0 > \cdots$$

be any path Z_0 in q for which y_0^0 is a maximal element. This is possible since q is a partially ordered set with no minimal element. Denote by A_0 the set

$$A_0 = \{y \mid y \in q,\ y \geqq y_\xi^0,\ y_\xi^0 \in Z_0\}.$$

We continue by transfinite induction. Suppose that the paths Z_μ and the sets A_μ have been defined for $\mu<\lambda$. Let y_0^λ be the first element of $q-\bigcup_{\mu<\lambda}A_\mu$. Let

$$y_0^\lambda > y_1^\lambda > \cdots > y_\xi^\lambda > \cdots$$

be any path Z_λ in q for which y_0^λ is a maximal element. This is possible since $q-\bigcup_{\mu<\lambda}A$ is a partially ordered set with no minimal element.

[4] A mapping g of a partially ordered set $(P, \leqq^1)$ into a partially ordered set $(Q, \leqq^2)$ is an isomorphism if g is one-to-one, and $p_1\leqq^1 p_2$ if and only if $g(p_1)\leqq^2 g(p_2)$.

Denote by A_λ the set

$$A_\lambda = \{y \mid y \in q,\ y \geqq \overset{\lambda}{y}_\xi,\ \overset{\lambda}{y}_\xi \in Z_\lambda\}.$$

Consider the two subsets of q

$$q_1 = \{\overset{\lambda}{y}_{\alpha+2n+1} \mid \alpha = 0 \text{ or a limit number, } \lambda \text{ any ordinal}\}$$

and

$$q_2 = \{\overset{\lambda}{y}_{\alpha+2n} \mid \alpha = 0 \text{ or a limit number, } \lambda \text{ any ordinal}\}.$$

Each of the two sets are coinitial in q. Therefore q_1 and q_2 are elements of Q.

Let g be the function which takes each point p of P into the element $g(p)$ of Q. Clearly g is an isomorphism of P into Q. Let h be the function which is defined by: (1) $h(q) = g(p_0)$ if p_0 is the unique maximal element of q; and (2) $h(q) = g(p_0)$, where p_0 is the first element of q, if q has no unique maximal element. If f is a real function on P, then denote by f_* the function which is defined by $f_*[g(p)] = p$ for p in P, and $f_*(q) = f_*[h(q)]$.

Suppose that f_* has a limit L on Q. For any element p_0 of P, and $\epsilon > 0$, denote by $E_\epsilon(p_0)$ the set

$$E_\epsilon(p_0) = \{p \mid p \leqq p_0 \text{ and } |f(p) - L| \geqq \epsilon\}.$$

If $E_\epsilon(p_0)$ were to be a coinitial subset of $g(p_0)$, then the relation $|f_*(q) - L| \geqq \epsilon$ would be true for all $q \leqq E_\epsilon(p_0)$. But this would contradict the function f_* having a limit L on Q. Therefore $E_\epsilon(p_0)$ is not a coinitial subset of $g(p_0)$. Consequently, for some element p_1 of $g(p_0)$, i.e., $p_1 \leqq p_0$, we have $\{p \mid p \leqq p_1\} \cap E_\epsilon(p_0) = \varnothing$. Hence $|f(p) - L| < \epsilon$ for $p \leqq p_1$. Thus f has a limit L on P.

Now suppose that the function f has a limit L on P. Let q_0 be any element of Q, and $\epsilon > 0$. Let p_0 be any element of q_0. For some element p_1 of P, where $p_1 \leqq p_0$, we have $|f(p) - L| < \epsilon$ for $p \leqq p_1$. If $q_1 = \{p \mid p \leqq p_1,\ p \in q_0\}$, then q_1 is a coinitial subset of $g(p_1)$. Thus q_1 is an element of Q. If $q \leqq q_1$, then $|f_*(q) - L| < \epsilon$. Therefore the function f_* has a limit L on Q.

When the partially ordered set P is directed, one can study the behavior on a coinitial subset of P of a real function f defined on P, by inspecting the everywhere branching partially ordered set Q and the function f_* which are obtained from the previous theorem. Specifically we have

THEOREM 2. *Let P be a directed partially ordered set and f a real function defined on it. Let Q and f_* be the same as in Theorem* 1.

Then a necessary and sufficient condition that f have a limit L on some coinitial subset q of P is that f_* have a partial limit L on Q.

PROOF. The necessity is trivial. Consider the sufficiency. Let Z be a residual subset of Q on which f_* has a limit L. Let q_0 be an element of Z. q_0 is a coinitial subset of P. Let $\epsilon > 0$ and p_0 be any element of q_0. Let

$$q_1 = \{p \mid p \leqq p_0, p \in q_0\}.$$

Suppose that for each point p_1 of q_1, a point p_2 of q_1, where $p_2 \leqq p_1$, can be found so that $|f(p_2) - L| \geqq \epsilon$. Since P is directed, and thus also q_0 and q_1,

$$q_2 = \{p \mid p \in q_1, |f(p) - L| \geqq \epsilon\}$$

is a coinitial subset of P. Furthermore, $|f_*(q) - L| \geqq \epsilon$ for $q \leqq q_1$. Thus f_* cannot have the limit L on Z. From this contradiction we see that for some $p_1 \leqq p_0$, $|f(p) - L| < \epsilon$ for $p \leqq p_1$. Consequently f has the limit L on q_0.

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