A UNIFORM APPROXIMATION TO FOURIER COEFFICIENTS

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Let $f(x)$ be a real-valued function which is continuous in the interval I: $-1 \leqq x \leqq 1$ except at N points at which it may have a finite jump. The sequence of its ordinary complex Fourier coefficients is given by

$$a_n = \frac{1}{2}\int_{-1}^{1} f(x)e^{-in\pi x}dx, \qquad n = 0, \pm 1, \cdots .$$

It is the purpose of this note to show that the elements of the sequence of Riemann sum approximations to a_n,

$$b_n(M) = \begin{cases} \dfrac{1}{2M} \displaystyle\sum_{j=-M+1}^{M} f\left(\frac{j}{M}\right) e^{-in\pi j/M}, & n = 0, \pm 1, \cdots, \pm M, \\ 0, & |n| > M \end{cases}$$

tend to a_n uniformly as $M \to \infty$.

Let

$$s_m = \sum_{j=-m}^{m} a_j e^{ij\pi x}, \qquad m = 0, 1, \cdots ,$$

and

$$\sigma_k(x) = \frac{1}{k+1}\sum_{m=0}^{k} s_m(x), \qquad k = 0, 1, \cdots .$$

Then

$$\sigma_k(x) = \sum_{n=-k}^{k} A_{k,n} e^{in\pi x}$$

where

$$A_{k,n} = \begin{cases} (1 - |n|/k + 1)a_n, & n = 0, \pm 1, \cdots, \pm k, \\ 0, & |n| > k. \end{cases}$$

Let us cover the set of points of discontinuity of $f(x)$ in I by a set E which is the sum of a finite number of open intervals of lengths η_i, $i = 1, 2, \cdots, N$, each of which covers at most one point of discontinuity and such that $\sum_{i=1}^{N} \eta_i \equiv \eta < \epsilon/6F$, where $F = \mathrm{LUB}|f(x)|$ in

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I. Then, on $I-IE$, by a generalization of Fejér's theorem on summability $(C, 1)$ of Fourier series (see, e.g., [1, p. 66]) $\sigma_k(x)$ converges uniformly to $f(x)$. Therefore, let M_1 be such that $|\sigma_k(x)-f(x)| < \epsilon/6$ uniformly on $I-IE$ for $k \geqq M_1$. Henceforth, let $M \geqq \max\{M_1, 3NF/\epsilon\}$. Since, in general, $|\sigma_k(x)| \leqq F$ (see [1, p. 58]), on IE we have $|\sigma_k(x)-f(x)| \leqq 2F$.

For $|n| \leqq M$, using the above inequalities, we obtain

$$\begin{aligned} |b_n(M) - A_{M,n}| &= \frac{1}{2M}\left|\sum_{j=-M+1}^{M}\left[f\left(\frac{j}{M}\right) - \sigma_M\left(\frac{j}{M}\right)\right]e^{-in\pi j/M}\right| \\ &\leqq \frac{1}{2M}\sum_{j=-M+1}^{M}\left|f\left(\frac{j}{M}\right) - \sigma_M\left(\frac{j}{M}\right)\right| \\ &\leqq \frac{1}{2M}\left[\frac{2M\epsilon}{6} + \sum_{i=1}^{N}(M\eta_i + 1)2F\right] < \frac{2\epsilon}{3}\cdot \end{aligned}$$

For $|n| > M$, $|b_n(M)-A_{M,n}| = 0$. For all n, we have

$$\begin{aligned} |A_{M,n} - a_n| &= \frac{1}{2}\left|\int_{-1}^{1}[\sigma_M(x) - f(x)]e^{-in\pi x}dx\right| \\ &< \frac{1}{2}\left(\frac{\epsilon}{3} + 2\eta F\right) < \frac{\epsilon}{3}\cdot \end{aligned}$$

Putting these results in the triangle inequality, we get $|b_n(M)-a_n| < \epsilon$ uniformly for all integral n.

Reference

1. Dunham Jackson, *The theory of approximation*, Amer. Math. Soc. Colloquium Publications, vol. 11, 1930.

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