THE FUNDAMENTAL GROUP OF THE PRINCIPAL COMPONENT OF A COMMUTATIVE BANACH ALGEBRA[1]

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We consider an arbitrary commutative Banach algebra over the complex numbers. Let B denote the algebra, $\{a, b, c, x, y, z, \cdots\}$ its elements, and $\{\lambda, \mu, \nu, \cdots\}$ complex numbers. We assume that B contains a unit element, e, with $\|e\| = 1$.

If a^{-1} exists ($aa^{-1} = a^{-1}a = e$), the element a is called "regular." The set of regular elements will be denoted by G. It is well known that (1) G is a topological group relative to multiplication and (2) G is an open subset of B. Since G is open, it is a union of maximal open connected sets, its components. We call the component G_1 containing the unit e the "principal component" [1]. It is easy to see that G_1 is a subgroup of G.

The function $\exp(x) \equiv e + \sum_1^\infty x^n/n!$ is defined for all x in B and has the usual properties of the classical exponential function. If we let $\pi_1(G_1)$ denote the fundamental group of G_1, we may state our main result as follows:

THEOREM 1. *Let $P = \{x \mid \exp(x) = e\}$. P is an additive group which is isomorphic to $\pi_1(G_1)$.*

We shall give a complete proof based on Schreier's theory of the universal covering group and then we shall outline a second proof which depends only on results from the theory of Banach algebras.[2] The result of Schreier which we shall use may be stated as follows [2]:

THEOREM. *Let B be a simply-connected, locally-connected and locally simply-connected topological group. If P is a discrete normal subgroup of B, then the fundamental group of the topological space B/P is isomorphic to the group P.*

The algebra B, regarded as an additive group with the metric topology of the norm, clearly satisfies the hypotheses of Schreier's

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[1] This result is contained in the author's doctoral dissertation, *The theory of analytic functions in Banach algebras*, completed in June 1952 under E. R. Lorch at Columbia University.

[2] The second proof is the one used in the author's dissertation. It was later pointed out by S. Eilenberg that a different proof is possible if one brings the Schreier theory to bear.

theorem. The set $P=\{x \mid \exp(x)=e\}$ is obviously a normal subgroup of B since $\exp(x-y)=\exp(x)\exp(-y)=\exp(x)[\exp(y)]^{-1}$. Further, $\exp(x)$ maps B onto the principal component, G_1 [3]. It is clearly a continuous map. It is also an open mapping since its inverse, $\log y$, is continuous. Thus, $\exp(x)$ is an open homomorphism of the additive topological group B onto the multiplicative topological group G_1. The kernel of this homomorphism is P, so that B/P is isomorphic to G_1 as topological groups. Hence $\pi_1(B/P)$ is isomorphic to $\pi_1(G_1)$. If it can be shown that P is discrete, the Schreier theorem is applicable and Theorem 1 follows immediately.

To prove P discrete it suffices to show that 0 is an isolated element of P. Suppose there is a sequence of elements $\{z_n\}$ such that $z_n \in P$, $z_n \neq 0$ and $\lim z_n=0$. By a theorem of Lorch [1], this implies that $z_n=2\pi i\sum_{j=1}^{k} n_j e_j$, where the e_j are idempotent elements ($e_j^2=e_j$) and the n_j are rational integers. Furthermore, the spectrum of z_n consists of the points $2\pi i n_1, \cdots, 2\pi i n_k$. However, $\lim \|z_n\|=0$ and, as is well known, the spectrum of z_n contains no points exterior to the circle of radius $\|z_n\|$, center at the origin. For sufficiently large n, this means that $n_j=0$ for $j=1, \cdots, k$; i.e. $z_n=0$. This contradiction completes the proof.

Now we shall indicate a more elementary and constructive proof in which no recourse is had to the Schreier theory.

First we establish a lemma concerning the function $\exp(x)$.

LEMMA: *Let* $w_0=\exp(x_0)$ *and* $0<\epsilon<1/\|w_0^{-1}\|$. *Let*

$$\delta = \sum_{1}^{\infty} \frac{(\|w_0^{-1}\|\epsilon)^n}{n}.$$

If E is the set $\{w \mid \|w-w_0\|<\epsilon\}$, then for every w in E there is an x such that $\|x-x_0\|<\delta$ and $\exp(x)=w$.

PROOF. Choosing $w=w_0+b$ where $\|b\|<\epsilon$, we have $\|w_0^{-1}w-e\| =\|w_0^{-1}b\| \leqq \|w_0^{-1}\|\|b\|<1$. Hence $w_0^{-1}w$ is in G_1 and there is an element c in B such that $\exp(c)=w_0^{-1}w$. In fact, we may take

$$c=\sum_{1}^{\infty} -(1/n)(e-w_0^{-1}w)^n$$

so that $\|c\|<\sum_1^\infty (1/n)(\|w_0^{-1}\|\epsilon)^n=\delta$. The element $x=x_0+c$ satisfies the conclusion of the lemma. It is important to note that δ approaches 0 as ϵ approaches 0.

Using this lemma, we are able to prove

THEOREM 2. *Let K: $\{f(s),\ 0\leqq s\leqq 1\}$ be a curve in G_1 joining $e=f(0)$*

to $w=f(1)$. *There exists an element* z *in* B *such that* $\exp(z)=w$ *and the curve* $K(z)$: $\{\exp(tz), 0\leqq t\leqq 1\}$ *is homotopic to* K *in* G_1.

PROOF. If z_s is such that $\exp(z_s)=f(s)$, let $K(z_s)$ denote the curve $\{\exp(tz_s), 0\leqq t\leqq 1\}$. $K(z_s)$ is a compact set in G_1. Hence, there is a number $\rho>0$ such that every sphere with center on $K(z_s)$ and radius ρ is contained in G_1. Choose ϵ such that $0<\epsilon<\min\{\rho, 1\}$. There is a number $\gamma(\epsilon)>0$ such that $\|f(r)-f(s)\|<\epsilon$ whenever $|r-s|<\gamma(\epsilon)$. By the lemma, there is an element z_r such that $\|z_r-z_s\|<\delta$ and $\exp(z_r)=f(r)$, where $\delta=\sum_1^\infty (1/n)(\|[f(s)]^{-1}\|\epsilon)^n$, that is, $z_r=z_s+b$ where $\|b\|<\delta$.

For all t, $0\leqq t\leqq 1$, $\|\exp(tz_r)-\exp(tz_s)\|\leqq\|\exp(tz_s)\|\cdot\|\exp(tb)-e\|\leqq\sum_0^\infty (1/n!)\|z_s\|^n\cdot\sum_1^\infty (1/n!)\|b\|^n\leqq\exp\|z_s\|\sum_1^\infty \delta^n/n!$. By choosing ϵ sufficiently small, thereby making δ small, we have $\|\exp(tz_r)-\exp(tz_s)\|<\rho$. It follows that $K(z_s)$ is homotopic in G_1 to the curve $K(z_r)\cup K_r^s$ consisting of $K(z_r)$ followed by the arc K_r^s: $\{f(u), r\leqq u\leqq s\}$.

In particular, since $\exp(0)=f(0)=e$, there is an $r>0$ and an element z_r such that $K(z_r)$ is homotopic to the arc K_0^r: $\{f(s)\mid 0\leqq s\leqq r\}$. The set of all real numbers r for which this holds has a least upper bound, $\mu\leqq 1$. Suppose $\mu<1$. We obtain a contradiction.

Let $\exp(z'_\mu)=f(\mu)$. There is an element z'_s such that $K(z'_\mu)$ is homotopic to $K(z'_s)\cup K_s^\mu$, $s<\mu$. But there is a z_s such that $K(z_s)$ is homotopic to K_0^s. Since $\exp(z'_s)=\exp(z_s)=f(s)$, we have $z_s=z'_s+c$ where $\exp(c)=e$. Let $z_\mu=z'_\mu+c$. It is simple to show that $K(z_\mu)$ is homotopic to $K(z_s)\cup K_s$ which is, in turn, homotopic to $K_0^s\cup K_s^\mu=K_0^\mu$. Hence $K(z_\mu)$ is homotopic to K_0^μ. By the same reasoning, there is a number $s_1>\mu$ such that $K_0^{s_1}$ is homotopic to $K(z_{s_1})$, contradicting the assumption on μ. Therefore, $\mu=1$.

The next theorem then follows easily.

THEOREM 3. *Every closed curve in* G_1 *with* e *as initial and end point is homotopic in* G_1 *to a curve* $K(b)$ *of the form* $\{\exp(tb)\mid 0\leqq t\leqq 1, \exp(b)=e\}$. *If* $\exp(b')=e$ *and* $b\neq b'$, *then* $K(b')$ *is not homotopic to* $K(b)$. *Thus each homotopy class contains precisely one curve of the form* $\{\exp(tb), 0\leqq t\leqq 1\}$ *where* $\exp(b)=e$.

PROOF. The first part of the theorem is obtained by choosing any point $w=f(s)$ on K. By Theorem 2, there is a z such that $K(z)$ is homotopic to K_0^s and a z' such that $K(z')$ is homotopic to K_s^1. The element $b=z'-z$ gives the desired result.

Noting that $\int_{K(b)}x^{-1}dx=b$ and $\int_{K(b')}x^{-1}dx=b'$, we see that $K(b)$ cannot be homotopic to $K(b')$, for in that event, the integrals would be equal by the Cauchy integral theorem (by [1] and a result in the author's dissertation not yet published).

Theorem 1 follows directly from Theorem 3.

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REMARK ON A FORMULA FOR THE BERNOULLI NUMBERS

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Some years ago Garabedian [1] proved the following formula:

$$B_{k+1} = \frac{(-1)^{k+1}(k+1)}{2^{k+1}-1} \sum_{r=0}^{k} (-1)^r \frac{\Delta^r 1^k}{2^{r+1}} \qquad (k \geqq 0), \tag{1}$$

where the even suffix notation is employed for the Bernoulli numbers. The proof of (1) made use of the sum of a certain divergent series.

We wish to point out that (1) is not new. It can be found (in somewhat different notation) in [3, p. 224, formula (68)].

It may be of interest to give a short proof of (1). We use the formula [2, p. 28]

$$C_k = 2^{k+1}(1 - 2^{k+1}) \frac{B_{k+1}}{k+1}, \tag{2}$$

where the C_k are the coefficients in the Euler polynomial:

$$E_k(x) = \left(x + \frac{C}{2}\right)^k = \sum_{s=0}^{k} \binom{k}{s} 2^{-s} C_s x^{k-s}. \tag{3}$$

Then in view of

$$E_k(x+1) + E_k(x) = 2x^k, \tag{4}$$

we have

$$E_k(x) = \left(1 + \frac{1}{2}\Delta\right)^{-1} x^k = \sum_{s=0}^{k} (-1)^s 2^{-s} \Delta^s x^k. \tag{5}$$

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