A NOTE ON THE DERIVATIONS OF LIE ALGEBRAS[1]

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1. **Introduction.** Let L be a Lie algebra and V a subalgebra of L. If W is another subalgebra of L such that $L = V + W$ and $V \cap W = (0)$, we say that *L splits over V* and that W is a *complement* of V in L. If V is an ideal in L, splitting of L over V is equivalent to the existence of an isomorphism θ of L/V into L such that $\phi\theta$ is the identity on L/V where ϕ is the natural homomorphism of L onto L/V.

A linear transformation T on the vector space of a Lie algebra L is called a *derivation* of L if $T(x \circ y) = T(x) \circ y + x \circ T(y)$ for all x, y in L. T is said to be an *inner derivation* of L if there exists an element y in L such that $T(x) = x \circ y$ for all x in L. Let T_1 and T_2 be two derivations of L. Then if we define their commutation $T_1 \circ T_2$ by $(T_1 \circ T_2)(x) = (T_2T_1 - T_1T_2)(x)$ for all x in L, it is easy to see that the derivations of L form a Lie algebra which we denote by $D(L)$. Further, it is easy to show that under this definition of commutation the inner derivations of L form an ideal in $D(L)$ which we denote by $I(L)$.

The purpose of this note is to prove the following theorem:

THEOREM. *Let L be a Lie algebra over a ground field of characteristic* 0 *and let R denote the radical of L. Then, if $D(R)$ splits over $I(R)$, $D(L)$ splits over $I(L)$.*

The converse of this theorem is still an open question, but in an appendix we give an example of a nilpotent Lie algebra whose derivations do not split over the inner derivations.

2. **Preliminaries.** Levi's well known decomposition theorem states that any Lie algebra L splits over its radical R and that any complement S of R must be a maximal semi-simple subalgebra of L. Malcev[2] has shown that any two complements of the radical are conjugate under an automorphism of L which has the property that it maps ideals of L which are contained in R onto themselves. Further, Harish-Chandra[3] has shown that any maximal semi-simple sub-

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[2] A. Malcev, C. R. Acad. Sci. URSS. vol. 36 (1942) p. 42.

[3] Harish-Chandra, *On the radical of a Lie algebra*, Proc. Amer. Math. Soc. vol. 1 (1950).

algebra of L is a complement of the radical. For our purposes we need the following easy extension of these results:

2.1. LEMMA. *Let R be the radical of L and let T be an ideal of L such that $T\subseteq R$ and such that L splits over T. Then if S is any semi-simple subalgebra of L, there exists a subalgebra Q of L with the properties:*

(1) *Q is a complement of T in L.*

(2) *$S\subseteq Q$.*

PROOF. Let Q' be any complement of T in L. Then since $L/R = (Q'+R)/R \cong Q'/Q'\cap R$, there exists a maximal semi-simple subalgebra $\mathfrak{S}'$ of L such that $\mathfrak{S}'\subseteq Q'$. Let $\mathfrak{S}$ be a maximal semi-simple subalgebra of L such that $S\subseteq\mathfrak{S}$. By the above results of Malcev and Harish-Chandra, there exists a conjugacy automorphism θ of L such that $\theta(\mathfrak{S}')=\mathfrak{S}$ and $\theta(T)=T$. Hence putting $Q=\theta(Q')$ gives the result.

Let $L=S+R$ as before and let $\mathfrak{A}(S)$ denote those derivations of L which map S into (0). It is easy to see that $\mathfrak{A}(S)$ is a subalgebra of $D(L)$ and that $\mathfrak{A}(S)\cap I(L)$ is an ideal of $\mathfrak{A}(S)$. Hochschild[4] has shown that $D(L)=\mathfrak{A}(S)+I(L)$ whence we have:

2.2. LEMMA. *If $\mathfrak{A}(S)$ splits over $\mathfrak{A}(S)\cap I(L)$, then $D(L)$ splits over $I(L)$.*

Notation. Throughout the remainder of this note, if x is any element of a Lie algebra X, x^* will denote the inner derivation of X induced by x. If Y is any subset of X, Y^* will denote the set of inner derivations of X induced by the elements of Y.

Now define a mapping ρ as follows: $\rho(D)$ is the restriction to R of D for all D in $D(L)$. Since the radical of a Lie algebra is characteristic with respect to derivations, ρ is a homomorphism of $D(L)$ into $D(R)$. Further, the restriction of ρ to $\mathfrak{A}(S)$ is an isomorphism of $\mathfrak{A}(S)$ into $D(R)$. The mapping $x\to x^*$ is clearly a homomorphism of L into $D(L)$. Thus since S is a semi-simple subalgebra of L, S^* is a semi-simple subalgebra of $D(L)$ and $\rho(S^*)$ is a semi-simple subalgebra of $D(R)$.

2.3. LEMMA. *$\rho(\mathfrak{A}(S))$ is the centralizer H, say, of $\rho(S^*)$ in $D(R)$.* [$H=\{T \mid T$ *in* $D(R);\ T\circ\rho(S^*)=(0)\}$.]

PROOF. If $\overline{T}$ is in $\rho(\mathfrak{A}(S))$, take T in $\mathfrak{A}(S)$ such that $\rho(T)=\overline{T}$.

$$T(r\circ s) = T(r)\circ s + r\circ T(s) = T(r)\circ s,$$

i.e. $\overline{T}(r\circ s)=\overline{T}(r)\circ s$ for all r in R, s in S. On the other hand,

[4] G. P. Hochschild, *Semi-simple algebras and generalized derivations*, Amer. J. Math. vol. 64 (1942).

$$\overline{T}(r \circ s) = \overline{T}(s^*(r)) = \overline{T}(\rho(s^*)(r)),$$
$$\overline{T}(r) \circ s = s^*(\overline{T}(r)) = \rho(s^*)(\overline{T}(r))$$

whence $\overline{T} \circ \rho(s^*) = 0$. Therefore $\rho(\mathfrak{A}(S)) \subseteq H$. Conversely, if $\overline{T}$ is in H, define a linear mapping T such that $T(s) = 0$, $T(r) = \overline{T}(r)$ for all s in S and all r in R. Then it is easy to see that T is a derivation and that $\rho(T) = \overline{T}$. Hence $H \subseteq \rho(\mathfrak{A}(S))$ and the lemma is proved.

2.4. LEMMA. $\rho(\mathfrak{A}(S) \cap I(L)) = \rho(\mathfrak{A}(S)) \cap I(R)$.

PROOF. Regard R as a representation space for S under the regular representation and let Z be the center of R. Since $S \circ Z \subseteq Z$ and since every representation of a semi-simple Lie algebra is semi-simple, we can find a vector subspace U of R such that $R = U + Z$, $U \cap Z = (0)$, and $S \circ U \subseteq U$. Now take T in $\rho(\mathfrak{A}(S)) \cap I(R)$ and let r_T be such that $T(r) = r \circ r_T$ for all r in R. Write $r_T = u + z$ with u in U and z in Z. Then $T(r) = r \circ u$ for all r in R. Since $T(S) = (0)$, we have $T(r \circ s) = T(r) \circ s$, i.e. $(r \circ s) \circ u = (r \circ u) \circ s$. Hence by the Jacobi identity, $(s \circ u) \circ r = 0$ for all r in R. Thus $s \circ u$ is in Z so that $s \circ u$ is in $Z \cap U = (0)$, whence $s \circ u = 0$. Therefore u^* is in $\mathfrak{A}(S) \cap I(L)$ and $\rho(u^*) = T$ proving that $\rho(\mathfrak{A}(S)) \cap I(R) \subseteq \rho(\mathfrak{A}(S) \cap I(L))$.

Conversely, take $\rho(x^*)$ in $\rho(\mathfrak{A}(S) \cap I(L))$. x^* in $\mathfrak{A}(S) \cap I(L)$ implies $x^*(s) = 0$ for all s in S, so that if we write $x = s_x + r_x$ with s_x in S and r_x in R, we have $0 = x^*(s) = s \circ s_x + s \circ r_x$ whence $s \circ s_x = r_x \circ s$ for all s in S. Hence $s_x = 0$ so that $x = r_x$ and $x^* = r_x^*$. The latter means that $\rho(x^*)$ is in $I(R)$ so that $\rho(\mathfrak{A}(S) \cap I(L)) \subseteq \rho(\mathfrak{A}(S)) \cap I(R)$ and the proof is complete.

If we remember that ρ is an isomorphism of $\mathfrak{A}(S)$ onto $\rho(\mathfrak{A}(S))$, then the above results include the following lemma:

2.5. LEMMA. $\mathfrak{A}(S)$ *splits over* $\mathfrak{A}(S) \cap I(L)$ *if and only if* $\rho(\mathfrak{A}(S))$ *splits over* $\rho(\mathfrak{A}(S)) \cap I(R)$.

3. **Proof of the main theorem.** Suppose that E is a complement of $I(R)$ in $D(R)$, i.e. $D(R) = E + I(R)$ and $E \cap I(R) = (0)$. Let $\mathfrak{R}$ be the radical of $D(R)$. Since R is solvable, $I(R)$ is a solvable ideal of $D(R)$ whence $I(R) \subseteq \mathfrak{R}$. Using the notation of §2 and applying Lemma 2.1 we may choose E so that $\rho(S^*) \subseteq E$.

Consider $D(R)$ as a $\rho(S^*)$ module in the regular representation. Clearly, $D(R) = E + I(R)$, direct sum as $\rho(S^*)$ modules (by the choice of E). Let X be the trivial $\rho(S^*)$ submodule of E and Y the trivial $\rho(S^*)$ submodule of $I(R)$. Then $X + Y$ is the trivial $\rho(S^*)$ submodule of $D(R)$. Therefore by Lemma 2.3 we have that $X + Y = \rho(\mathfrak{A}(S))$. Since $Y \subseteq I(R)$, we have $(X + Y) \cap I(R) = X \cap I(R) + Y = Y$. Hence

$Y=\rho(\mathfrak{A}(S))\cap I(R)$. We show next that X is a subalgebra of $\rho(\mathfrak{A}(S))$. Take T_1, T_2 in X, then

$$\begin{aligned}\rho(s^*)\cdot(T_1\circ T_2) &= (T_1\circ T_2)\circ\rho(s^*)\\ &= -(T_2\circ\rho(s^*))\circ T_1 - (\rho(s^*)\circ T_1)\circ T_2\\ &= -(\rho(s^*)\cdot T_2)\circ T_1 + (\rho(s^*)\cdot T_1)\circ T_2 = 0,\end{aligned}$$

so that $T_1\circ T_2$ is in $\rho(\mathfrak{A}(S))$. But $X\subseteq E$ and E is a subalgebra of $D(R)$ so that $T_1\circ T_2$ is in E and hence is in X.

Thus X is a complement of $\rho(\mathfrak{A}(S))\cap I(R)=Y$ in $\rho(\mathfrak{A}(S))$ so that, by Lemma 2.5, $\mathfrak{A}(S)$ splits over $\mathfrak{A}(S)\cap I(L)$ and the theorem follows by Lemma 2.2.

3.1. Corollary. *If the radical of L is abelian, then $D(L)$ splits over $I(L)$.*

4. Appendix. We now give an example of a nilpotent Lie algebra L which is such that $D(L)$ does not split over $I(L)$.[5]

Let L have a basis x_1, x_2, x_3, x_4 over a ground field K where (x_2, x_3, x_4) is abelian and $x_1\circ x_2=x_4$, $x_1\circ x_3=0$, $x_1\circ x_4=0$. Clearly $L\circ L=(x_4)$ and the center of L is (x_3, x_4). $I(L)$ consists of derivations of the form $k_1x_1^*+k_2x_2^*$ and we note that $(k_1x_1^*+k_2x_2^*)(x_1)=k_2x_4$, $(k_1x_1^*+k_2x_2^*)(x_2)=-k_1x_4$. Thus if D_1 is any derivation of L we may add an inner derivation to it so that the resulting derivation has the form:

$$\begin{aligned}D(x_1) &= ax_1+bx_2+cx_3, & D(x_2) &= dx_1+ex_2+fx_3,\\ D(x_3) &= gx_3+hx_4, & D(x_4) &= (a+e)x_4.\end{aligned}$$

Note that no such derivation can be in $I(L)$.

Suppose now that $D(L)=E+I(L)$ is a splitting of $D(L)$ over $I(L)$ and consider the derivations D_1, D_2 such that $D_1(x_1)=x_3$, $D_1(x_i)=0$ if $i\neq 1$; $D_2(x_3)=x_4$, $D_2(x_i)=0$ if $i\neq 3$. Then $D_1\circ D_2=x_2^*$. Write $D_1=e_1+r_1^*$, $D_2=e_2+r_2^*$ with e_1, e_2 in E, and $r_1=k_1x_1+k_2x_2$, $r_2=k_1'x_1+k_2'x_2$. Then

$$\begin{aligned}e_1\circ e_2 &= (D_1-r_1^*)\circ(D_2-r_2^*) = D_1\circ D_2 - D_1\circ r_2^*\\ &\quad + D_2\circ r_1^* - r_1^*\circ r_2^*\\ &= x_2^* + (D_1(r_2))^* - (D_2(r_1))^* - (r_1\circ r_2)^* = x_2^*,\end{aligned}$$

i.e. $e_1\circ e_2$ is in $E\cap I(L)$ so that $e_1\circ e_2=0=x_2^*$, which is a contradiction.

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[5] This example was communicated to me in a letter by Professor Hochschild.