A REMARK ON ZETA FUNCTIONS

H. KOBER

1. Let \( s = \sigma + it \), \( \xi(s) = \sum_{n=1}^{\infty} n^{-s} (\sigma > 1) \), \( \omega(x) = \sum_{n=1}^{\infty} e^{-nx} (x > 0) \),

\[
\xi(s) = \pi^{-s/2} \Gamma(s/2) \xi(s) = \int_{0}^{\infty} \omega(x)x^{\sigma - 1}dx \quad (\sigma > 1).
\]

Then \( s(s-1)\xi(s) \) is well known to be an entire function, and its zeros are identical with the nontrivial zeros of \( \xi(s) \), i.e. with those lying in the strip \( 0 < \sigma < 1 \). Furthermore let

\( w = u + iv, w \neq 0; |\arg w|, |\arg 1/w|, |\arg (x+w)| \leq \pi/2; s \neq 0,1 \),

be fixed;

\[
F_s(w) = \int_{0}^{\infty} \omega(x+w)(x+w)^{\sigma - 1}dx - \frac{w^{(e-1)/2}}{1-s}.
\]

Then \( F_s(w) \) is an analytic function of \( w \) for \( u > 0 \), since

\[
|\omega(x+w)| \leq x^{-1/2}e^{1-x} \quad (x > 0);
\]

its limit function \( F_s(\infty) \) exists for any \( v \geq 0 \) by the Lebesgue convergence theorem. Now we can deduce that

\[
F_s(w) + F_{1-s}(1/w) = \xi(s) \quad (u \geq 0, w \neq 0).
\]

For \( v = 0 \) this reduces to the, possibly known, equation

\[
(1.3a) \quad \xi(s) = \int_{0}^{\infty} \omega(x)x^{\sigma - 1}dx + \int_{1/u}^{\infty} \omega(x)x^{(1+s)/3}dx - \frac{u^{(e-1)/2}}{1-s} - \frac{u^{s/2}}{s}.
\]

Hence (1.3) hold by analytic continuation. Clearly \( F_s(w) \rightarrow \xi(s)(w \rightarrow 0; 0 < \sigma < 1) \).

Again (1.3) takes simple forms for \( w = i \) and \( w = 2i \):

\[
(1.3b) \quad \xi(s) = \int_{0}^{\infty} \lambda(x)(x+i)^{\sigma - 1}dx
\]

\[
+ \int_{0}^{\infty} \lambda(x)(x-i)^{-(\sigma+1)/3}dx - \frac{e^{i\pi(s-1)/4}}{1-s} - \frac{e^{i\pi s/4}}{s};
\]

Received by the editors June 2, 1952 and, in revised form, November 7, 1952.

\(^1\) For \( u = 1 \) this is the classical equation due to Riemann. E.g. E. Landau, Handbuch der Lehre von der Verteilung der Primzahlen, Leipzig and Berlin, 1909, §70; by a similar argument (1.3a) is deduced.
A REMARK ON ZETA FUNCTIONS

\[ \xi(s) = \int_0^\infty \omega(x)(x + 2i)^{s/2 - 1}dx \]

\[ + \int_0^\infty \left\{ i\omega(x) + (1 - i)\omega(4x) \right\} \left( x - \frac{i}{2} \right)^{-(s+1)/2} dx \]

\[ - \frac{(2e^{i\pi/2})^{(s-1)/2}}{1 - s} - \frac{(2e^{i\pi/2})^{s/2}}{s} ; \]

where \( \omega(x) = \sum (-1)^n e^{-\pi n^2 x}. \) By the formula \( |\Gamma(s)|e^{\pi s/2}l^{1/2-s} \to \) constant \((0 < t \to \infty)\) the Lindelöf hypothesis is equivalent to the statement

\[ \Re \left\{ \int_0^\infty \omega(x)(x + i)^{-3/4+i/2} dx \right\} = O(t^{-1/4+e^{-\pi/14}}) \]

\((\epsilon > 0 \text{ fixed}; 0 < t \to \infty).\)

2. The following theorems, easily derived from (1.3), give criteria for the nontrivial zeros of \( \xi(s) \):

Theorem 1. A given point \( s \) is a nontrivial zero of \( \xi(s) \) if and only if, for \( w \neq 0 \) with \( u \geq 0 \), \( F_u(w) \) satisfies the functional equation

\[ F_u(w) = - F_{1-s}(1/w). \]

Theorem 2. Let \( 0 < \sigma < 1 \). Then \( s \) is a zero of \( \xi(s) \) if and only if

\[ \omega^{(t-1)/2} \int_0^\infty (x + w)^{-(s+1)/2} \left\{ \frac{1}{2} (x + w)^{-1/2} - \omega(x + w) \right\} dx \]

\[ \to \frac{1}{s - 1} \quad (u \geq 0; |w| \to 0), \]

or, for any fixed \( a > 0 \) (for instance, for \( a = 1 \)),

\[ \int_0^a x^{s-1/2 - 1} \left\{ \frac{1}{2} x^{-1/2} - \rho^{1/2} \omega(x/\rho) \right\} dx \to \frac{a^{(t-1)/2}}{s - 1} \quad (0 < \rho \to \infty). \]

If \( s \) is not a zero, the moduli of the terms on the left of (i) and (ii) tend to infinity.

Remark. For fixed \( s \) \((0 < \sigma < 1)\), \( F_u(w) \sim w^{(s-1)/2(s-1)-1} \to 0 \)

\((|w| \to \infty)\); \( F_u(w) \) is bounded and uniformly continuous \((u \geq 0)\); \( F_u(w) = (2\pi)^{-1} \int_{-\infty}^\infty dx F_u(i\alpha)(w - i\alpha)^{-1} \quad (u > 0)\); and\(^{3}\) \( \xi(s) = i/\pi \). PV.

\(^{3}\) The assertion, still unproved, that \( \xi(1/2 + it) = O(t^\epsilon) \quad (\epsilon > 0; 0 < t \to \infty)\). This is known to be equivalent to \( \int_{-\infty}^{\infty} e^{-\pi x^2/\lambda(x)} dx = O(t^{-1/4+e^{-\pi/14}})\).

\(^{3}\) PV. \( \int_{-\infty}^{\infty} = \lim_{a \to \infty} \left( \int_{-a}^{a} + \int_{a}^{\infty} \right) \) is the "principal value" of the integral. The above representations of \( F_u(w) \) and \( \xi(s) \) by integrals follow from the theory of the Hille-Tamarkin class \( S_\delta \); see Fund. Math. vol. 25 (1935) pp. 329–352.
Beyond the line \( m = 0 \) the function \( F_z(w) \) cannot be continued analytically.

3. In a recent paper T. M. Apostol has investigated the functional equation of the generalized zeta function \( \phi(s, a, b) = \sum_0^\infty e^{2\pi i n b}(n+a)^{-s} \), due to Lerch, for the case \( b \to 1 \) \((0 < a \leq 1, 0 < b < 1)\). The problem can be considerably simplified and, incidentally, generalized. Replace \( \phi(s, a, b) \) by \( \xi(s, a, b) \) and introduce \( Z_1(s, a, b), Z_2(s, a, b) \), where \( a \) and \( b \) are any real numbers,

\[
\xi(s, a, b) = \sum_{n>\alpha} e^{2\pi i n b}(n + a)^{-s},
\]

\[
Z_1(s, a, b) = \sum_{n=-\infty, n+a=0}^\infty \frac{e^{2\pi i n b}}{|n + a|},
\]

\[
Z_2(s, a, b) = \sum_{n=-\infty, n+a=0}^\infty \frac{e^{2\pi i n b}(n + a)}{|n + a|^{s+1}} \quad (\sigma < 1).
\]

Obviously

\[
2\xi(s, a, b) = Z_1(s, a, b) + Z_2(s, a, b);
\]

\[
2\xi(s, -a, -b) = Z_1(s, a, b) - Z_2(s, a, b),
\]

and we obtain the functional equations

\[
e^{2\pi i n b}\chi(s + k - 1)Z_k(s, a, b) = i^{k-1}\chi(k - s)Z_k(1 - s, b, -a) \quad (k = 1, 2),
\]

\[
\frac{(2\pi)^s}{\Gamma(s)} e^{2\pi i a b} \zeta(1 - s, a, b)
\]

\[
e^{\pi i s / 2} \zeta(s, b, -a) + e^{-\pi i s / 2} \zeta(s, -b, a),
\]

where \( \chi(s) = \zeta(s, a, b) \) is deduced from well known formulae on theta series, by the classical method; while, by (3.1), (3.3) is a corollary of it.

**BIRMINGHAM, ENGLAND**


\* E.g. H. Kober, J. Reine Angew. Math. vol. 174 (1936) pp. 206–225, §4. Again the equation (3.2) for \( Z_k(s, a, b) \) is deduced by Apostol in the special case \( 0 < a < 1 \), Pacific Journal of Mathematics vol. 1 (1951) pp. 161–167. For his function \( \Lambda(x, a, s) \), defined for \( 0 < a < 1 \) and treated by the classical method (see pp. 161–163), reduces to \( Z_1(s, a, x) \), etc. as is easily shown.

\* I.e. \( \theta(x, a, b) = e^{2\pi i n b} x^{-1/2}(x^2 - 1, b, -a) \) and the formula gained from this by differentiation with respect to \( a \); where \( \theta(x, a, b) = \sum_{-\infty}^\infty \exp \{ -\pi x(n+a)^2 + 2\pi i n b \} = \theta(x, -a, -b) \).