A REMARK ON ZETA FUNCTIONS

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1. Let $s=\sigma+it$, $\zeta(s)=\sum_1^\infty n^{-s}$ $(\sigma>1)$, $\omega(x)=\sum_{n=1}^\infty e^{-n^2\pi x}$ $(x>0)$,

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^\infty \omega(x)x^{s/2-1}dx \qquad (\sigma > 1). \tag{1.1}$$

Then $s(s-1)\xi(s)$ is well known to be an entire function, and its zeros are identical with the nontrivial zeros of $\zeta(s)$, i.e. with those lying in the strip $0<\sigma<1$. Furthermore let

$w = u + iv$, $w \neq 0$; $|\arg w|$, $|\arg 1/w|$, $|\arg(x + w)| \leqq \pi/2$; $s \neq 0,1$,

be fixed;

$$F_s(w) = \int_0^\infty \omega(x + w)(x + w)^{s/2-1}dx - \frac{w^{(s-1)/2}}{1-s}. \tag{1.2}$$

Then $F_s(w)$ is an analytic function of w for $u>0$, since

$$|\omega(x + w)| \leqq x^{-1/2}e^{1-x} \qquad (x > 0);$$

its limit function $F_s(iv)$ exists for any $v\gtrless 0$ by the Lebesgue convergence theorem. Now we can deduce that

$$F_s(w) + F_{1-s}(1/w) = \xi(s) \qquad (u \geqq 0,\ w \neq 0). \tag{1.3}$$

For $v=0$ this reduces to the, possibly known, equation[1]

$$\xi(s) = \int_u^\infty \omega(x)x^{s/2-1}dx + \int_{1/u}^\infty \omega(x)x^{-(1+s)/2}dx - \frac{u^{(s-1)/2}}{1-s} - \frac{u^{s/2}}{s}. \tag{1.3a}$$

Hence (1.3) hold by analytic continuation. Clearly $F_s(w)\to\xi(s)$ $(w\to 0;$ $0<\sigma<1)$.

Again (1.3) takes simple forms for $w=i$ and $w=2i$:

$$\begin{aligned}\xi(s) = &\int_0^\infty \lambda(x)(x + i)^{s/2-1}dx \\ &+ \int_0^\infty \lambda(x)(x - i)^{-(s+1)/2}dx - \frac{e^{i\pi(s-1)/4}}{1-s} - \frac{e^{i\pi s/4}}{s};\end{aligned} \tag{1.3b}$$

Received by the editors June 2, 1952 and, in revised form, November 7, 1952.

[1] For $u=1$ this is the classical equation due to Riemann. E.g. E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, Leipzig and Berlin, 1909, §70; by a similar argument (1.3a) is deduced.

$$
\begin{aligned}
\xi(s) = &\int_0^\infty \omega(x)(x+2i)^{s/2-1}dx \\
&+ \int_0^\infty \{i\omega(x) + (1-i)\omega(4x)\}\left(x - \frac{i}{2}\right)^{-(s+1)/2} dx \\
&- \frac{(2e^{i\pi/2})^{(s-1)/2}}{1-s} - \frac{(2e^{i\pi/2})^{s/2}}{s};
\end{aligned} \tag{1.3c}
$$

where $\lambda(x) = \sum_1^\infty (-1)^n e^{-n^2\pi x}$. By the formula $|\Gamma(s)| e^{\pi t/2} t^{1/2-\sigma} \to$ constant $(0 < t \to \infty)$ the Lindelöf hypothesis[2] is equivalent to the statement

$$
\Re\left\{\int_0^\infty \lambda(x)(x+i)^{-3/4+it/2}dx\right\} = O(t^{-1/4+\epsilon}e^{-\pi t/4})
$$

$$
(\epsilon > 0 \text{ fixed}; 0 < t \to \infty).
$$

2. The following theorems, easily derived from (1.3), give criteria for the nontrivial zeros of $\zeta(s)$:

THEOREM 1. *A given point s is a nontrivial zero of $\zeta(s)$ if and only if, for $w \neq 0$ with $u \geqq 0$, $F_s(w)$ satisfies the functional equation*

$$
F_s(w) = -F_{1-s}(1/w).
$$

THEOREM 2. *Let $0 < \sigma < 1$. Then s is a zero of $\zeta(s)$ if and only if*

$$
\begin{aligned}
&w^{(s-1)/2}\int_0^\infty (x+w)^{-(s+1)/2}\left\{\frac{1}{2}(x+w)^{-1/2} - \omega(x+w)\right\}dx \\
&\qquad \to \frac{1}{s-1} \qquad (u \geqq 0;\ |w| \to 0),
\end{aligned} \tag{i}
$$

or, for any fixed $a > 0$ (for instance, for $a = 1$),

$$
\int_0^a x^{s/2-1}\left\{\frac{1}{2}x^{-1/2} - \rho^{1/2}\omega(x\rho)\right\}dx \to \frac{a^{(s-1)/2}}{s-1} \qquad (0 < \rho \to \infty). \tag{ii}
$$

If s is not a zero, the moduli of the terms on the left of (i) *and* (ii) *tend to infinity.*

REMARK. For fixed s $(0 < \sigma < 1)$, $F_s(w) \sim w^{(s-1)/2}(s-1)^{-1} \to 0$ $(|w| \to \infty)$; $F_s(w)$ is bounded and uniformly continuous $(u \geqq 0)$; $F_s(w) = (2\pi)^{-1}\int_{-\infty}^{\infty} d\alpha F_s(i\alpha)(w - i\alpha)^{-1}$ $(u > 0)$; and[3] $\xi(s) = i/\pi$ PV.

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[2] The assertion, still unproved, that $\zeta(1/2+it) = O(t^\epsilon)$ $(\epsilon > 0; 0 < t \to \infty)$. This is known to be equivalent to $\int_0^\infty x^{-3/4+it/2}\lambda(x)dx = O(t^{-1/4+\epsilon}e^{-\pi t/4})$.

[3] PV. $\int_{-\infty}^{\infty} = \lim_{\epsilon\to 0}\left(\int_{-\infty}^{-\epsilon} + \int_\epsilon^\infty\right)$ is the "principal value" of the integral. The above representations of $F_s(w)$ and $\xi(s)$ by integrals follow from the theory of the Hille-Tamarkin class $\mathfrak{H}_p$; see Fund. Math. vol. 25 (1935) pp. 329–352.

$\int_{-\infty}^{\infty} d\alpha F_s(i\alpha)/\alpha$. Beyond the line $u=0$ the function $F_s(w)$ can not be continued analytically.

3. In a recent paper T. M. Apostol[4] has investigated the functional equation of the generalized zeta function $\phi(s, a, b) = \sum_0^\infty e^{2\pi i n b}(n+a)^{-s}$, due to Lerch, for the case $b \to 1$ $(0 < a \leq 1,\ 0 < b < 1)$. The problem can be considerably simplified and, incidentally, generalized. Replace $\phi(s, a, b)$ by $\zeta(s, a, b)$ and introduce $Z_1(s, a, b)$, $Z_2(s, a, b)$, where a and b are *any real numbers*,

$$\zeta(s, a, b) = \sum_{n > -a} e^{2\pi i n b}(n + a)^{-s},$$

$$Z_1(s, a, b) = \sum_{n=-\infty, n+a\neq 0}^{\infty} \frac{e^{2\pi i n b}}{|n + a|^s},$$

$$Z_2(s, a, b) = \sum_{n=-\infty, n+a\neq 0}^{\infty} \frac{e^{2\pi i n b}(n + a)}{|n + a|^{s+1}} \qquad (\sigma < 1).$$

Obviously

$$\begin{aligned} 2\zeta(s, a, b) &= Z_1(s, a, b) + Z_2(s, a, b); \\ 2\zeta(s, -a, -b) &= Z_1(s, a, b) - Z_2(s, a, b), \end{aligned} \tag{3.1}$$

and we obtain the functional equations

$$e^{2\pi i a b}\chi(s + k - 1)Z_k(s, a, b) = i^{k-1}\chi(k - s)Z_k(1 - s, b, -a) \qquad (k = 1, 2), \tag{3.2}$$

$$\frac{(2\pi)^s}{\Gamma(s)} e^{2\pi i a b}\zeta(1 - s, a, b) = e^{\pi i s/2}\zeta(s, b, -a) + e^{-\pi i s/2}\zeta(s, -b, a), \tag{3.3}$$

where $\chi(s) = \pi^{-s/2}\Gamma(s/2)$; a, b real. The equation (3.2), known in special cases,[5] is deduced from well known formulae on theta series,[6] by the classical method; while, by (3.1), (3.3) is a corollary of it.

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[4] *Remark on the Hurwitz zeta function*, Proceedings of the American Mathematical Society vol. 2 (1951) pp. 690–693.

[5] E.g. H. Kober, J. Reine Angew. Math. vol. 174 (1936) pp. 206–225, §4. Again the equation (3.2) for $Z_k(s, a, b)$ is deduced by Apostol in the special case $0 < a < 1$, Pacific Journal of Mathematics vol. 1 (1951) pp. 161–167. For his function $\Lambda(x, a, s)$, defined for $0 < a < 1$ and treated by the classical method (see pp. 161–163), reduces to $Z_1(s, a, x)$, etc. as is easily shown.

[6] I.e. $\theta(x, a, b) = e^{-2\pi i a b}x^{-1/2}\theta(x^{-1}, b, -a)$ and the formula gained from this by differentiation with respect to a; where $\theta(x, a, b) = \sum_{-\infty}^{\infty} \exp\{-\pi x(n+a)^2 + 2\pi i n b\} = \theta(x, -a, -b)$.