NOTE ON SOME PARTITION IDENTITIES

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1. **Introduction.** In a recent paper, Newman [4] states the formulas

$$(1.1)\qquad \sum_0^\infty p_2(11m+10)x^n = \sum_1^\infty (1-x^{11n})^2,$$

$$(1.2)\qquad \sum_0^\infty p_4(11m+20)x^n = -11\prod_1^\infty (1-x^{11n})^4,$$

$$(1.3)\qquad \sum_0^\infty p_2(17m+24)x^n = -\prod_1^\infty (1-x^{17n})^2,$$

$$(1.4)\qquad \sum_0^\infty p_6(31m+240)x^n = 961\prod_1^\infty (1-x^{31n})^6,$$

where

$$\prod_{n=1}^\infty (1-x^n)^k = \sum_{m=0}^\infty p_k(m)x^m.$$

We wish to point out that results of this kind can be obtained in a very elementary way, namely, by using a method employed by Ramanujan in proving the formula $p(5m+4)\equiv 0 \pmod 5$ (see for example [2, p. 87]). We shall prove the following formulas. Let r be prime. If $r\equiv 3 \pmod 4$, $r>3$, then

$$(1.5)\qquad \sum_{m=0}^\infty p_2(rm+r_0)x^m = \prod_{n=1}^\infty (1-x^{rn})^2,$$

where $r_0=(r^2-1)/12$.

If $r\equiv 3 \pmod 4$, $r\geqq 3$, then

$$(1.6)\qquad \sum_{m=0}^\infty p_6(rm+r_1)x^m = r^2\prod_{n=1}^\infty (1-x^{rn})^6,$$

where $r_1=(r^2-1)/4$.

If $r\equiv 5 \pmod 6$, then

$$(1.7)\qquad \sum_{m=0}^\infty p_4(rm+r_2)x^m = -r\prod_{n=1}^\infty (1-x^{rn})^4,$$

where $r_2=(r^2-1)/6$.

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If $r\equiv 5 \pmod{12}$, then

$$(1.8)\qquad \sum_{m=0}^{\infty} p_2(rm + r_0)x^m = -\prod_{n=1}^{\infty}(1 - x^{rn})^2,$$

where $r_0=(r^2-1)/12$.

It is clear that (1.1) is contained in (1.5), (1.2) in (1.7), (1.3) in (1.8), (1.4) in (1.6); the case $r=5$ of (1.7) occurs in [3]. We also remark that (1.5), $\cdots$, (1.8) can be put in somewhat sharper form; for example in place of (1.5) we can state

$$\sum_{m=0}^{\infty} p_2(r^2m + r_0)x^m = \prod_{n=1}^{\infty}(1 - x^n)^2 = \sum_{m=0}^{\infty} p_2(m)x^m.$$

In other words

$$p_2(r^2m + r_0) = p_2(m); \qquad p_2(rm + r_0) = 0 \quad \text{for} \quad r\nmid m.$$

Similar results hold for the other functions.

2. **Proof of (1.5).** By Euler's formula

$$(2.1)\qquad x^s\prod_{n=1}^{\infty}(1 - x^n)^2 = \sum_{h,k=-\infty}^{\infty}(-1)^{h+k}x^{s+h(3h+1)/2+k(3k+1)/2},$$

where s is to be assigned. The exponent on the right is divisible by r provided

$$(2.2)\qquad (6h + 1)^2 + (6k + 1)^2 + 2(12s - 1) \equiv 0 \pmod{r}.$$

If we take s as the least positive integer such that $12s\equiv 1 \pmod{r}$, then by the hypothesis on r it is clear that (2.2) implies $r\mid 6h+1$, $r\mid 6k+1$. Thus with a little manipulation (2.1) yields

$$\sum_{m=0}^{\infty} p_2(rm + r - s)x^m = x^e\prod_{n=1}^{\infty}(1 - x^{rn})^2,$$

where

$$(2.3)\qquad e = \frac{12s - 1}{12r} + \frac{r}{12} - 1.$$

Since

$$re + r - s = \frac{12s - 1}{12} + \frac{r^2}{12} - s = \frac{r^2 - 1}{12},$$

(1.5) follows at once.

3. **Proof of (1.6).** Using Jacobi's formula we have

$$x^s\prod_{n=1}^{\infty}(1-x^n)^6 \\ = \sum_{h,k=0}^{\infty}(-1)^{h+k}(2h+1)(2k+1)x^{s+h(h+1)/2+k(k+1)/2}. \tag{3.1}$$

The exponent on the right is divisible by r provided

$$(2h+1)^2+(2k+1)^2+2(4s-1)\equiv 0 \pmod{r}. \tag{3.2}$$

If we choose s as the least positive integer such that $4s\equiv 1 \pmod{r}$, (3.2) implies $r\mid 2h+1$, $r\mid 2k+1$. Thus, very much as above, (3.1) yields

$$\sum_{m=0}^{\infty}p_6(rm+r-s)x^m = r^2x^e\prod_{n=1}^{\infty}(1-x^{rn})^6,$$

where

$$e=\frac{8s-1}{8r}+\frac{r}{4}-1.$$

Since

$$re+r-s=\frac{8s-1}{8}+\frac{r^2}{4}-s=\frac{r^2-1}{4},$$

(1.6) follows at once.

4. **Proof of (1.7).** Using Euler's and Jacobi's formula we have

$$x^s\prod_{n=1}^{\infty}(1-x^n)^4=\frac{1}{2}\sum_{h,k=-\infty}^{\infty}(-1)^{h+k}(2k+1)x^{s+h(3h+1)/2+k(k+1)/2}. \tag{4.1}$$

The exponent on the right is divisible by r provided

$$(6h+1)^2+3(2k+1)^2+4(6s-1)\equiv 0 \pmod{r}. \tag{4.2}$$

We choose s as the least positive integer such that $6s\equiv 1 \pmod{r}$. Since -3 is a quadratic nonresidue of r, it follows from (4.2) that $r\mid 6h+1$, $r\mid 2k+1$. A little attention must now be paid to the sign in the right member of (4.1). We find without much trouble that (4.1) implies

$$\sum_{m=0}^{\infty}p_4(rm+r-s)x^m = -rx^e\prod_{n=1}^{\infty}(1-x^{rn})^4, \tag{4.3}$$

where

$$e = \frac{s-1}{6r} + \frac{r}{6} - 1.$$

Since

$$re + r - s = \frac{s-1}{6} + \frac{r^2}{6} - s = \frac{r^2-1}{6},$$

it is evident that (4.3) reduces to (1.7).

5. **Proof of (1.8).** We return to (2.1) and (2.2). Since $r \equiv 1 \pmod 4$ we can no longer assert that $r \mid 6h+1$, $r \mid 6k+1$, but only that $(6h+1)^2 + (6k+1)^2 \equiv 0 \pmod p$. Changing the notation slightly, consider

$$(5.1) \qquad h = au - bv, \qquad k = av + bu,$$

where $r = a^2 + b^2$ and $h \equiv k \equiv 1 \pmod 6$. Since $r \equiv 5 \pmod{12}$, we may suppose that $a \equiv 1$, $b \equiv \pm 2 \pmod 6$. If $b \equiv 2 \pmod 6$, consider

$$(5.2) \quad rh' = -(a^2 - b^2)h - 2abk, \quad rk' = -2abh + (a^2 - b^2)k.$$

Then by (5.1), (5.2) reduces to $h' = -au - bv$, $k' = -bu + av$, so that h' and k' are integers; moreover $h'^2 + k'^2 = h^2 + k^2$. In the next place (5.2) implies

$$5h' \equiv 3h - 4k \equiv -1, \qquad h' \equiv 1 \pmod 6,$$
$$5k' \equiv -4h + 3k = -1, \quad k' \equiv 1.$$

On the other hand (5.2) implies

$$(5.3) \qquad h' \equiv -h, \qquad k' \equiv k \pmod 4.$$

It follows that the terms in the right member of (2.1) corresponding to (h, k) and (h', k') cancel.

Next, if $b \equiv -2 \pmod 6$, we change all signs in the right members of (5.2). The details are much as before; in particular (5.3) becomes $h' \equiv h$, $k' \equiv -k \pmod 4$. Thus once again corresponding terms cancel.

Now consider a pair (h, k) with $h^2 + k^2 = m$, where m is fixed, $r \mid m$, $h \equiv k \equiv 1 \pmod 6$. Suppose first $r \nmid h$. Then if $r \nmid h'$, it is clear from the above that the corresponding terms in (2.1) cancel. On the other hand, when $r \mid h$, then it follows from the above discussion that we can simultaneously consider the correspondence (5.2) together with the second correspondence ($b \equiv -2$). In other words we have in this case ($r \mid h$) a (2, 1) correspondence. Returning to (2.1) we see that

$$\sum_{m=0}^{\infty} p_2(rm + r - s) = -x^e \prod_{n=1}^{\infty} (1 - x^{rn})^2,$$

where e is determined by (2.3). The proof of (1.8) is now completed in exactly the same way as in (1.5).

6. **Another formula.** Newman also states the formula

$$\sum_{m=0}^{\infty} p_6(5m)x^m = \prod_{n=1}^{\infty} (1 - x^n)^6(1 - x^{5n})^{-1}, \tag{6.1}$$

which he notes had been found (but not published) by D. H. Lehmer. It may be of interest to point out that (6.1) can be obtained easily from the identity.

$$\prod_{n=1}^{\infty} \frac{(1 - x^n)^5}{1 - x^{5n}} = 1 - 5\sum_{m=1}^{\infty}\left(\frac{m}{5}\right)\frac{x^m}{1 - x^m}. \tag{6.2}$$

The formula (6.2) is due to Ramanujan; Bailey [1] showed recently that it is a consequence of well known formulas for the Weierstrass elliptic functions.

Since the right member of (6.2) equals

$$1 - 5 \sum_{m,r=1}^{\infty} \left(\frac{m}{5}\right) m x^{mr},$$

it follows that

$$\sum_{m=0}^{\infty} p_6(5m)x^{5m}\prod_{n=1}^{\infty} (1 - x^{5n})^{-1} = 1 - 5\sum_{m=1}^{\infty}\left(\frac{m}{5}\right)\frac{x^{5m}}{1 - x^{5m}}$$
$$= \prod_{n=1}^{\infty} (1 - x^{5n})^5(1 - x^{25n})^{-1}.$$

Replacing x^5 by x we get (6.1).

References

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