NOTE ON SOME PARTITION IDENTITIES

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1. Introduction. In a recent paper, Newman [4] states the formulas

\begin{align*}
(1.1) & \quad \sum_{0}^{\infty} p_2(11m + 10)x^n = \sum_{1}^{\infty} (1 - x^{11n})^2, \\
(1.2) & \quad \sum_{0}^{\infty} p_3(11m + 20)x^n = -11 \prod_{1}^{\infty} (1 - x^{11n})^4, \\
(1.3) & \quad \sum_{0}^{\infty} p_5(17m + 24)x^n = - \prod_{1}^{\infty} (1 - x^{17n})^2, \\
(1.4) & \quad \sum_{0}^{\infty} p_6(31m + 240)x^n = 961 \prod_{1}^{\infty} (1 - x^{31n})^4,
\end{align*}

where

\[ \prod_{n=1}^{\infty} (1 - x^n)^k = \sum_{m=0}^{\infty} p_k(m)x^m. \]

We wish to point out that results of this kind can be obtained in a very elementary way, namely, by using a method employed by Ramanujan in proving the formula \( p(5m+4) \equiv 0 \pmod{5} \) (see for example [2, p. 87]). We shall prove the following formulas. Let \( r \) be prime. If \( r \equiv 3 \pmod{4} \), \( r > 3 \), then

\begin{align*}
(1.5) & \quad \sum_{m=0}^{\infty} p_2(rm + r_0)x^n = \prod_{n=1}^{\infty} (1 - x^{rn})^2, \\
\text{where } r_0 & = (r^2 - 1)/12.
\end{align*}

If \( r \equiv 3 \pmod{4} \), \( r \equiv 3 \), then

\begin{align*}
(1.6) & \quad \sum_{m=0}^{\infty} p_3(rm + r_1)x^n = r^2 \prod_{n=1}^{\infty} (1 - x^{rn})^4, \\
\text{where } r_1 & = (r^2 - 1)/4.
\end{align*}

If \( r \equiv 5 \pmod{6} \), then

\begin{align*}
(1.7) & \quad \sum_{m=0}^{\infty} p_6(rm + r_2)x^n = -r \prod_{n=1}^{\infty} (1 - x^{rn})^4, \\
\text{where } r_2 & = (r^2 - 1)/6.
\end{align*}

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If \( r \equiv 5 \pmod{12} \), then

\[
(1.8) \quad \sum_{m=0}^{\infty} p_2(rm + r_0)x^m = - \prod_{n=1}^{\infty} (1 - x^{rn})^2,
\]

where \( r_0 = (r^2-1)/12 \).

It is clear that (1.1) is contained in (1.5), (1.2) in (1.7), (1.3) in (1.8), (1.4) in (1.6); the case \( r = 5 \) of (1.7) occurs in [3]. We also remark that (1.5), \ldots, (1.8) can be put in somewhat sharper form; for example in place of (1.5) we can state

\[
\sum_{m=0}^{\infty} p_2(r^2m + r_0)x^m = \prod_{n=1}^{\infty} (1 - x^n)^2 = \sum_{m=0}^{\infty} p_2(m)x^m.
\]

In other words

\[
p_2(r^2m + r_0) = p_2(m); \quad p_2(rm + r_0) = 0 \quad \text{for} \quad r \nmid m.
\]

Similar results hold for the other functions.

2. Proof of (1.5). By Euler's formula

\[
(2.1) \quad x^s \prod_{n=1}^{\infty} (1 - x^n)^2 = \sum_{h, k = -\infty}^{\infty} (-1)^{h+k} x^{h^2 + (2h+1)/2 + k(3k+1)/2},
\]

where \( s \) is to be assigned. The exponent on the right is divisible by \( r \) provided

\[
(2.2) \quad (6h + 1)^2 + (6k + 1)^2 + 2(12s - 1) \equiv 0 \pmod{r}.
\]

If we take \( s \) as the least positive integer such that \( 12s \equiv 1 \pmod{r} \), then by the hypothesis on \( r \) it is clear that (2.2) implies \( r \mid 6h+1, \ r \mid 6k+1 \). Thus with a little manipulation (2.1) yields

\[
\sum_{m=0}^{\infty} p_2(rm + r - s)x^m = x^e \prod_{n=1}^{\infty} (1 - x^{rn})^2,
\]

where

\[
e = \frac{12s - 1}{12} + \frac{r}{12} - 1.
\]

Since

\[
re + r - s = \frac{12s - 1}{12} + \frac{r^2}{12} - s = \frac{r^2 - 1}{12},
\]

(1.5) follows at once.
3. Proof of (1.6). Using Jacobi's formula we have

\[ x^r \prod_{n=1}^{\infty} (1 - x^n)^6 \]

\[ \sum_{h,k=0}^{\infty} (-1)^{h+k}(2h+1)(2k+1)x^{r^4} \left( \frac{r^2-1}{4} \right)^{k-1} \]

The exponent on the right is divisible by \( r \) provided

\[ (2h+1)^2 + (2k+1)^2 + 2(4s - 1) \equiv 0 \pmod{r}. \]

If we choose \( s \) as the least positive integer such that \( 4s \equiv 1 \pmod{r} \), (3.2) implies \( r \mid 2h+1, r \mid 2k+1 \). Thus, very much as above, (3.1) yields

\[ 0 \sum_{m=0}^{\infty} p_s(rm + r - s)x^m = r^2x^r \prod_{n=1}^{\infty} (1 - x^n)^6, \]

where

\[ e = \frac{8s - 1}{8r} + \frac{r}{4} - 1. \]

Since

\[ re + r - s = \frac{8s - 1}{8} + \frac{r^2}{4} - s = \frac{r^2 - 1}{4}, \]

(1.6) follows at once.

4. Proof of (1.7). Using Euler's and Jacobi's formula we have

\[ x^s \prod_{n=1}^{\infty} (1 - x^n)^4 = \frac{1}{2} \sum_{h,k=0}^{\infty} (-1)^{h+k}(2h+1)x^{r^4} \left( \frac{r^2-1}{2} \right)^{k-1} \]

The exponent on the right is divisible by \( r \) provided

\[ (6h+1)^2 + 3(2k+1)^2 + 4(6s - 1) \equiv 0 \pmod{r}. \]

We choose \( s \) as the least positive integer such that \( 6s \equiv 1 \pmod{r} \). Since \( -3 \) is a quadratic nonresidue of \( r \), it follows from (4.2) that \( r \mid 6h+1, r \mid 2k+1 \). A little attention must now be paid to the sign in the right member of (4.1). We find without much trouble that (4.1) implies

\[ 0 \sum_{m=0}^{\infty} p_s(rm + r - s)x^m = -r x^r \prod_{n=1}^{\infty} (1 - x^n)^4, \]

where
\[ e = \frac{s - 1}{6r} + \frac{r}{6} - 1. \]

Since
\[ re + r - s = \frac{s - 1}{6} + \frac{r^2}{6} - s = \frac{r^2 - 1}{6}, \]
it is evident that (4.3) reduces to (1.7).

5. Proof of (1.8). We return to (2.1) and (2.2). Since \( r \equiv 1 \pmod{4} \) we can no longer assert that \( r \mid 6k + 1, r \mid 6k + 1 \), but only that \( (6k + 1)^2 + (6k + 1)^2 \equiv 0 \pmod{p} \). Changing the notation slightly, consider
\[
(5.1) \quad h = au - bv, \quad k = av + bu,
\]
where \( r = a^2 + b^2 \) and \( h \equiv k \equiv 1 \pmod{6} \). Since \( r \equiv 5 \pmod{12} \), we may suppose that \( a \equiv 1, b \equiv \pm 2 \pmod{6} \). If \( b \equiv 2 \pmod{6} \), consider
\[
(5.2) \quad rh' = -(a^2 - b^2)h - 2abk, \quad rk' = -2abk + (a^2 - b^2)k.
\]
Then by (5.1), (5.2) reduces to \( h' = -au - bv, k' = -bu + av \), so that \( h' \) and \( k' \) are integers; moreover \( h'^2 + k'^2 = h^2 + k^2 \). In the next place (5.2) implies
\[
5h' \equiv 3h - 4k \equiv -1, \quad h' \equiv 1 \pmod{6},
\]
\[
5k' \equiv -4h + 3k = -1, \quad k' \equiv 1.
\]
On the other hand (5.2) implies
\[
(5.3) \quad h' \equiv -h, \quad k' \equiv k \pmod{4}.
\]
It follows that the terms in the right member of (2.1) corresponding to \( (h, k) \) and \( (h', k') \) cancel.

Next, if \( b \equiv -2 \pmod{6} \), we change all signs in the right members of (5.2). The details are much as before; in particular (5.3) becomes \( h' \equiv h, k' \equiv -k \pmod{4} \). Thus once again corresponding terms cancel.

Now consider a pair \((h, k)\) with \( h^2 + k^2 = m \), where \( m \) is fixed, \( r \mid m \), \( h \equiv k \equiv 1 \pmod{6} \). Suppose first \( r \mid h \). Then if \( r \mid h' \), it is clear from the above that the corresponding terms in (2.1) cancel. On the other hand, when \( r \mid h \), then it follows from the above discussion that we can simultaneously consider the correspondence (5.2) together with the second correspondence \( (b = -2) \). In other words we have in this case \( r \mid h \) a (2, 1) correspondence. Returning to (2.1) we see that
\[
\sum_{m=0}^{\infty} p_{r}(rm + r - s) = -x^{r} \prod_{n=1}^{\infty} (1 - x^{rn})^{2},
\]
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where \( e \) is determined by (2.3). The proof of (1.8) is now completed in exactly the same way as in (1.5).

6. Another formula. Newman also states the formula

\[
\sum_{m=0}^{\infty} \rho_6(5m)x^m = \prod_{n=1}^{\infty} (1 - x^n)^6(1 - x^{5n})^{-1},
\]

which he notes had been found (but not published) by D. H. Lehmer. It may be of interest to point out that (6.1) can be obtained easily from the identity.

\[
\prod_{n=1}^{\infty} \frac{(1 - x^n)^6}{1 - x^{5n}} = 1 - 5 \sum_{m=1}^{\infty} \left(\frac{m}{5}\right) \frac{x^m}{1 - x^m}.
\]

The formula (6.2) is due to Ramanujan; Bailey [1] showed recently that it is a consequence of well known formulas for the Weierstrass elliptic functions.

Since the right member of (6.2) equals

\[
1 - 5 \sum_{m=1}^{\infty} \left(\frac{m}{5}\right) m x^m,
\]

it follows that

\[
\sum_{m=0}^{\infty} \rho_6(5m) x^m \prod_{n=1}^{\infty} (1 - x^{5n})^{-1} = 1 - 5 \sum_{m=1}^{\infty} \left(\frac{m}{5}\right) \frac{x^m}{1 - x^{5m}} = \prod_{n=1}^{\infty} (1 - x^{5n})^6(1 - x^{25n})^{-1}.
\]

Replacing \( x^6 \) by \( x \) we get (6.1).

REFERENCES

2. G. H. Hardy, Ramanujan, Cambridge, 1940.

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