# A CARDINAL NUMBER ASSOCIATED WITH A FAMILY OF SETS[1]

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Let $U$ be a family of nonempty subsets of an abstract set $R$, partially ordered by set inclusion.[2] The smallest cardinal number which is the power of the union of a maximal family of incomparable elements of $U$ shall be defined as the "maximal density" of $U$ (md $(U)$). The smallest cardinal number which is the md $(V)$ of some coinitial subfamily $V$ of $U$ shall be defined as the "containing maximal density" of $U$ (cmd $(U)$). The principal result of this paper is Theorem 2, which states that $\prod_{\xi<\alpha}$ cmd $(U^\xi)=$cmd $(\prod_{\xi<\alpha} U^\xi)$.

Before turning to our main result we consider the containing maximal density of a ramified family of sets.[3]

THEOREM 1. *If $U$ is a ramified family of sets, then the maximal density of $U$ equals the containing maximal density of $U$.*

PROOF. Let $V=\{E\}$ be a coinitial subfamily of $U=\{D\}$, such that md $(V)=$cmd $(U)$. Let $M=\{H\}$ be a maximal family of incomparable elements of $V$ for which md $(V)=p(M)$.[4] We shall now show that $M$ is a maximal family of incomparable elements of $U$. Let $D$ be any element in $U$, and $E$ any element of $V$ which is a subset of $D$. The element $E$ certainly exists since $V$ is a coinitial subset of $U$. The family $M$ being a maximal family of incomparable elements of $V$, there exists an element $H$ in $M$ which is comparable with $E$. If $H$ is a subset of $E$, then $H$ is also a subset of $D$. Suppose that $E$ is a subset of $H$. Since $U$ is ramified, it follows that the two elements $D$ and $H$ are comparable. Consequently each element of $U$ is comparable with some element in $M$. Thus $M$ is a maximal family of incomparable elements in $U$. Therefore

$$\text{md } (U) \leqq \text{md } (V) = \text{cmd } (U) \leqq \text{md } (U).$$

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[2] In the sequel $U$ and $U^\xi$ will be a family of subsets of the sets $R$ and $R^\xi$ respectively.

[3] The family of sets $U$ is a ramified family of sets if, for each element $E$ in $U$, the family of sets $\{D \mid D \supseteq E, D \in U\}$ is monotone.

[4] Let $Y$ be a family of sets. By $p(Y)$ is meant the power of the set which is the set union of the elements in $Y$. If $Y$ consists of only one set, say $A$, then $p(Y)$ is the power of $A$. In this case, $p(A)$ is also used, i.e., $p(A)=p(Y)$.

Thus $\mathrm{md}(U) = \mathrm{cmd}\ (U)$.

In preparation for Theorem 2, two lemmas are needed.

LEMMA 1. *The containing maximal density of $U$ is the smallest cardinal number equal to $p(V)$, for some coinitial subfamily $V$ of $U$.*

PROOF. Let $V = \{E\}$ be any coinitial subfamily of $U$ for which $\mathrm{cmd}\ (U) = \mathrm{md}\ (V)$. Let $M = \{E_\nu\}$ be a maximal family of incomparable elements of $V$ such that $p(M) = \mathrm{md}\ (V)$. Now the subfamily of $V$,

$$Y = \{E \mid E \subseteq E_\nu,\ E \in V,\ E_\nu \in M, \text{ for some } \nu\},$$

is coinitial in $V$. Thus $p(Y) \leqq \mathrm{md}\ (V) = \mathrm{cmd}\ (U)$. Now let $p(Z)$ be the smallest cardinal number for some coinitial subfamily $Z$ of $U$. Clearly $\mathrm{cmd}\ (U) \leqq \mathrm{md}\ (Z) \leqq p(Z)$. This completes the proof.

LEMMA 2. *To each family $U = \{D\}$, there corresponds a coinitial subfamily $V = \{E\}$, and a subfamily, $Y = \{D_\nu \mid \nu < \delta\}$ of $V$, which have the following properties:*

(1) *each element of $V$ is $p$-homogeneous;*[5]

(2) $\{E \mid E \subseteq D_\xi,\ E \in V\} \cap \{E \mid E \subseteq D_\nu,\ E \in V\} = \varnothing$ *for* $\xi \neq \nu$;

(3) *if $\{G_\xi \mid \xi < \delta\}$ is any subfamily of $V$ in which $G_\xi$ is a subset of $D_\xi$ for each $\xi$, then*

$$p\left(G_\nu - \bigcup_{\xi<\nu} D_\xi\right) = p\left(D_\nu - \bigcup_{\xi<\nu} D_\xi\right) \qquad (\nu < \delta),$$

(4) $p[\bigcup_{\nu<\delta} (D_\nu - \bigcup_{\xi<\nu} D_\xi)] \geqq \mathrm{cmd}\ (V)$.

PROOF. If $V$ is a coinitial subfamily of $U$ such that $\mathrm{md}\ (V) = \mathrm{cmd}\ (U)$, and

$$Z = \{E \mid E \in V,\ E \text{ is } p\text{-homogeneous}\},$$

then $Z$ is a coinitial subfamily of $U$ for which $\mathrm{md}\ (Z) = \mathrm{cmd}\ (U)$. In order to simplify the notation, it is assumed that $U$ has the two properties of $Z$, i.e., (a) $\mathrm{md}\ (U) = \mathrm{cmd}\ (U)$, and ($b$) each element $D$ in $U$ is $p$-homogeneous. Well order the elements of $U$, $D_0$ being the first element. Suppose that the family of sets $\{D_\xi \mid \xi < \lambda\}$ has already been defined. Denote by $D_\lambda$ the first element $D_*$ in $U$ which satisfies the following two conditions:

(c) if $D$ is a subset of $D_*$, where $D$ is in $U$, then $D$ is not a subset of $\bigcup_{\xi<\lambda} D_\xi$;

[5] An element $E$ in $V$ is $p$-homogeneous if $p(B) = p(E)$ for each element $B$ in $V$ which is a subset of $E$. See Erdös and Tarski, *On families of mutually exclusive sets*, Ann. of Math. vol. 44 (1943) pp. 315–329.

(d) if $D$ is a subset of $D_*$, where $D$ is in $U$, then $p(D-\bigcup_{\xi<\lambda} D_\xi)=p(D_*-\bigcup_{\xi<\lambda} D_\xi)$. If the set of elements satisfying (c) is nonempty, then the element $D_\lambda$ certainly exists. Let $\{D_\xi \mid \xi<\delta\}$ be a maximal family obtained in this way.

Let $V=\{E\}$ be the subfamily of $U$,

$$V=\left\{D \mid D\subseteq \bigcup_{\xi<\delta} D_\xi,\, D\in U\right\}.$$

Conditions (1), (2), (3), and (4) are automatically satisfied. It is necessary to show only that $V$ is a coinitial subfamily of $U$. Suppose that $D_*$ is an element of $U$ which contains no element of $V$. Then the element $D_*$ satisfies condition (c). This implies that the family $\{D_\xi \mid \xi<\delta\}$ is not maximal. From this contradiction we obtain our conclusion.

We now prove our main result.

THEOREM 2. cmd $(\prod_{\xi<\alpha} U^\xi)=\prod_{\xi<\alpha}$ cmd $(U^\xi)$, *where by* $\prod_{\xi<\alpha} U^\xi$ *is meant the cartesian product of the* $U^\xi$.[6]

PROOF. To each family $U^\xi$, associate a family of sets, $Y^\xi=\{D^\xi_\gamma \mid \gamma<\delta_\xi\}$, which satisfies the conclusions of Lemma 2. Let $U$ be a coinitial subfamily of $\prod_\xi U^\xi$ for which $p(U)=\text{md}\,(U)=\text{cmd}\,(U)$. In the following, $\beta$, $\gamma$, and $\xi$ denote indices that range over specified sets of ordinal numbers, whereas $\mu$ and $\nu$ denote indices that range over the class of all $\alpha$-sequences $\sigma=\{\sigma_\xi\}$, $\xi<\alpha$, where $\sigma_\xi<\delta_\xi$ for each $\xi$. The notation $\mu<\nu$ refers to the partial ordering defined by

$$\mu<\nu \text{ if and only if } \begin{cases} \mu_\xi \leqq \nu_\xi & \text{for all } \xi<\alpha, \text{ and} \\ \mu_\xi<\nu_\xi & \text{for some } \xi<\alpha. \end{cases}$$

For each $\nu$, $\prod_\xi D^\xi_{\nu_\xi}$ belongs to $\prod_\xi U^\xi$. Since $U$ is a coinitial subfamily of $\prod_\xi U^\xi$, there exists an element $D_\nu$ in $U$ such that $D_\nu \subseteq \prod_\xi D^\xi_{\nu_\xi}$. $D_\nu$ is of the form $D_\nu=\prod_\xi E^\xi_\nu$, where for each $\xi<\alpha$, $E^\xi_\nu$ is an element of $U^\xi$ and $E^\xi_\nu\subseteq D^\xi_{\nu_\xi}$. It now follows that

$$\begin{aligned}\text{cmd}\left(\prod_\xi U^\xi\right) &= p(U) \geqq p\left(\bigcup_\nu D_\nu\right) \\ &\geqq p\left[\bigcup_\nu \left(D_\nu - \bigcup_{\mu<\nu} D_\nu\right)\right] \geqq p\left[\bigcup_\nu\left(D_\nu - \bigcup_{\mu<\nu}\prod_\xi D^\xi_{\mu_\xi}\right)\right].\end{aligned}$$

Since each two terms inside the last bracket are disjoint,

[6] The cartesian product of the $U^\xi$ is the family of sets $\{H \mid H=D^0\times D^1\times\cdots,\ D^\xi\in U^\xi\}$.

$$p\left[\bigcup_{\nu}\left(D_\nu - \bigcup_{\mu<\nu}\prod_{\xi} D^{\xi}_{\mu_\xi}\right)\right]$$
$$= \sum_{\nu} p\left(D_\nu - \bigcup_{\mu<\nu}\prod_{\xi} D^{\xi}_{\mu_\xi}\right)$$
$$= \sum_{\nu} p\left(\prod_{\xi} E^{\xi}_\nu - \bigcup_{\mu<\nu}\prod_{\xi} D^{\xi}_{\mu_\xi}\right),$$

which, since $\prod_\xi (E^\xi_\nu - \bigcup_{\beta<\nu_\xi} D^\xi_\beta) \subseteq (\prod_\xi E^\xi_\nu - \bigcup_{\mu<\nu}\prod_\xi D^\xi_{\mu_\xi})$, is

$$\geqq \sum_{\nu} p\left[\prod_{\xi}\left(E^{\xi}_\nu - \bigcup_{\beta<\nu_\xi} D^{\xi}_{\beta}\right)\right]$$
$$= \sum_{\nu}\prod_{\xi} p\left(E^{\xi}_\nu - \bigcup_{\beta<\nu_\xi} D^{\xi}_{\beta}\right)\Big]$$
$$= \sum_{\nu}\prod_{\xi} p\left(D^{\xi}_{\nu_\xi} - \bigcup_{\beta<\nu_\xi} D^{\xi}_{\beta}\right) \qquad \text{(by condition 3)}$$
$$= \prod_{\xi}\sum_{\gamma<\delta_\xi} p\left(D^{\xi}_{\gamma} - \bigcup_{\beta<\gamma} D^{\xi}_{\beta}\right) \qquad \text{(by the distributive law)}$$
$$= \prod_{\xi} p\left[\bigcup_{\gamma<\delta_\xi}\left(D^{\xi}_{\gamma} - \bigcup_{\beta<\gamma} D^{\xi}_{\beta}\right)\right]$$
$$\geqq \prod_{\xi} \mathrm{cmd}\,(U^\xi).$$

Thus cmd $(\prod_\xi U^\xi) \geqq \prod_\xi$ cmd $(U^\xi)$. Since the reverse inequality is obviously true, it follows that $\prod_\xi$ cmd $(U^\xi)=$ cmd $(\prod_\xi U^\xi)$.

For each $\xi<\mu$ let $U^\xi$ be a family of infinite subsets of a set $R^\xi$. By $\prod'_{\xi<\mu} U^\xi$ we shall mean the family of sets

$\{G \mid G=\prod H^\xi$, where $H^\xi \in U^\xi$, or $H^\xi = R^\xi$, and all but a finite number of the $H^\xi$ are $R^\xi\}$.

THEOREM 3. *cmd* $(\prod'_\xi U^\xi)$ *is the smallest cardinal number, call it* $\aleph_\beta$, *which is in the set of cardinal numbers* $\{\aleph_\alpha \mid \aleph_\alpha =$ cmd $(U^{\alpha(0)} \times U^{\alpha(1)} \times \cdots \times U^{\alpha(n)}) \cdot \prod_{\xi\neq\alpha(j), j\leqq n} p(R^\xi);\, n<\omega\}$.

PROOF. If $V^\xi$ is a coinitial subfamily of $U^\xi$, then cmd $(\prod' U^\xi)$ $=$ cmd $(\prod' V^\xi)$. Thus, no generality is lost in assuming that for each $\xi$

$$p(U^\xi) = \mathrm{md}\,(U^\xi) = \mathrm{cmd}\,(U^\xi).$$

Furthermore, it may be assumed that

$$\aleph_\beta = \text{cmd } (U^0 \times \cdots \times U^n) \cdot \prod_{\xi > n} p(R^\xi).$$

To prove the theorem, it is sufficient to show that $\aleph_\beta \leqq \text{cmd } (\prod' U^\xi)$. Let $U$ be a coinitial subfamily of $\prod' U^\xi$ for which $p(U) = \text{md } (U) = \text{cmd } (\prod' U^\xi)$, and let $D$ be any element of $U$. To simplify the notation, suppose that

$$D = D^0 \times \cdots \times D^{n+m} \times \prod_{\xi > n+m} R^\xi.$$

Then

$$\begin{aligned} \text{cmd } (\textstyle\prod' U^\xi) &= \text{md } (U) = p(U) \geqq p(D) \\ &= p(D^0 \times D^1 \times \cdots \times D^{n+m}) \cdot \prod_{\xi > n+m} p(R^\xi) \\ &\geqq \prod_{\xi > n+m} p(R^\xi). \end{aligned}$$

Also,

$$\begin{aligned} \text{cmd } (\textstyle\prod' U^\xi) &\geqq \text{cmd } (U^0 \times \cdots \times U^{n+m}) \\ &= \prod_{\xi \leqq n+m} \text{cmd } (U^\xi) \qquad \text{(Theorem 2)} \\ &= \prod_{\xi \leqq n+m} p(U^\xi) = p\left( \prod_{\xi \leqq n+m} U^\xi \right). \end{aligned}$$

Combining our results we get

$$\begin{aligned} \aleph_\beta &\leqq p\left( \prod_{j \leqq n+m} U^j \right) \cdot p\left( \prod_{\xi > n+m} R^\xi \right) \\ &\leqq \text{cmd}\left( \prod_\xi{}' U^\xi \right) \cdot \text{cmd}\left( \prod_\xi{}' U^\xi \right) \\ &= \text{cmd } (\textstyle\prod' U^\xi), \end{aligned}$$

the last equality resulting from the fact that as each element of $U^\xi$ is infinite, cmd $(\prod' U^\xi)$ is infinite.

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