ON EULER METHODS OF SUMMABILITY FOR DOUBLE SERIES

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The two qth order Euler transforms of the sequence A_n

$$A_n^q = (q+1)^{-n-1}\sum_{k=0}^{n}\binom{n+1}{k+1}q^{n-k}A_k$$

and

$$B_n^q = (q+1)^{-n}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}A_k$$

are equivalent for $q \geqq 0$ in the sense that if either has a limit as $n \to \infty$ the other has the same limit [1, p. 180].[1] For double sequences the corresponding transforms are

$$(1)\qquad A_{mn}^q = (q+1)^{-m-n-2}\sum_{h,k=0}^{m,n}\binom{m+1}{h+1}\binom{n+1}{k+1}q^{m+n-h-k}A_{hk},$$

$$(2)\qquad B_{mn}^q = (q+1)^{-m-n}\sum_{h,k=0}^{m,n}\binom{m}{h}\binom{n}{k}q^{m+n-h-k}A_{hk}.$$

This paper is concerned with two theorems regarding these transforms. Throughout the discussion $q \geqq 0$.

THEOREM 1. *If A_{mn}^q has a limit as $m, n \to \infty$, then B_{mn}^q has that same limit and if B_{mn}^q has a limit and is bounded, then A_{mn}^q has that same limit but there do exist sequences for which B_{mn}^q has a limit but for which* $\lim_{m,n\to\infty} A_{mn}^q$ *does not exist for any $q \geqq 0$.*

The relation

$$(3)\qquad B_{mn}^q = q^2A_{m-1,n-1}^q - q(q+1)(A_{m,n-1}^q + A_{m-1,n}^q) + (q+1)^2A_{mn}^q$$

may be verified by substitution from (1) into the right-hand side. This relation may be written in the form

$$(4)\qquad \begin{aligned} B_{mn}^q = {} & q^2(A_{m-1,n-1}^q - A_{m,n-1}^q - A_{m-1,n}^q + A_{mn}^q) \\ & - q(A_{m,n-1}^q + A_{m-1,n}^q - 2A_{mn}^q) + A_{mn}^q. \end{aligned}$$

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[1] Numbers in brackets refer to the references at the end of the paper.

From this relation (4) it follows that if $\lim_{m,n\to\infty} A^q_{mn}=A$, then $\lim_{m,n\to\infty} B^q_{mn}=A$.

Relation (3) can be used to express A^q_{mn} in terms of B^q_{mn}. First write (3) in the form

$$(q+1)^2 A^q_{mn} = B^q_{mn} + q(q+1)(A^q_{m,n-1} + A^q_{m-1,n}) - q^2 A^q_{m-1,n-1}.$$

In this replace $A^q_{m,n-1}$ and $A^q_{m-1,n}$, by the values which this relation gives for them. This yields

$$\begin{aligned}(q+1)^2 A^q_{mn} = B^q_{mn} &+ q(q+1)^{-1}(B^q_{m,n-1} + B^q_{m-1,n})\\ &+ q^2(A^q_{m,n-2} + A^q_{m-1,n-1} + A^q_{m-2,n})\\ &- q^3(q+1)^{-1}(A^q_{m-1,n-2} + A^q_{m-2,n-1}).\end{aligned}$$

Successive repetitions of this procedure lead finally to the relation

$$(5)\qquad (q+1)^2 A^q_{mn} = \sum_{h,k=0}^{m,n} \left(\frac{q}{q+1}\right)^{m+n-h-k} B^q_{hk}.$$

Relation (5) expresses A^q_{mn} as a transform of the sequence B^q_{mn}. The coefficients of the transformation satisfy the conditions for regularity [3, p. 23]. Hence if B^q_{mn} has the limit A and is bounded, then A^q_{mn} also has the limit A.

To see that there exist sequences for which the transform B^q_{mn} has a limit but for which $\lim_{m,n\to\infty} A^q_{mn}$ does not exist for any $q\geqq 0$ consider the sequence $A_{mn}=(-1)^n p^{2m+n-1}\{n(p+1)+p\}$, $p>1$. For this sequence one may readily verify by substitution into (2) that $B^p_{mn}=0$ whenever $n>1$. Thus for this sequence the transform B^p_{mn} has the limit 0. But by substituting into (1) and simplifying one obtains

$$\begin{aligned}A^q_{mn} = &\left\{\left(\frac{q+p^2}{q+1}\right)^{m+1} - \left(\frac{q}{q+1}\right)^{m+1}\right\}\\ &\cdot\left(\frac{n+1}{q+1}\right)\cdot\left(\frac{p+1}{p^3}\right)\cdot\left(\frac{q-p}{q+1}\right)^n\\ &+ p^{-3}\left\{\left(\frac{q+p^2}{q+1}\right)^{m+1} - \left(\frac{q}{q+1}\right)^{m+1}\right\}\\ &\left\{\left(\frac{q-p}{q+1}\right)^{n+1} - \left(\frac{q}{q+1}\right)^{n+1}\right\}\end{aligned}$$

and for $p>1$ this does not have a limit for any $q\geqq 0$. This completes the proof of Theorem 1.

THEOREM 2. *Let* $A_{mn} = \sum_{h,k=0}^{m,n} a_{hk}$. If

$$(1)\ \overset{1}{B}_{mn} = 2^{-m-n} \sum_{h,k=0}^{m,n} \binom{m}{h}\binom{n}{k} A_{hk}$$

has the limit A *as* $m, n \to \infty$, (2) A_{mn} *is bounded and*

$$(3)\quad \lim_{m,n\to\infty} (m^{1/2} + n^{1/2})(mn)^{1/2} a_{mn} = 0,$$

then A_{mn} *also has the limit* A *as* $m, n \to \infty$.

Form the difference

$$\overset{1}{B}_{4m,4n} - A_{2m,2n} = 2^{-4m-4n} \sum_{h,k=0}^{4m,4n} \binom{4m}{h}\binom{4n}{k} (A_{hk} - A_{2m,2n}).$$

Separate this difference into 9 parts $S_1, S_2, \cdots, S_9$ corresponding respectively to the intervals of summation

$$\begin{pmatrix} 0 \leqq h \leqq m \\ 0 \leqq k \leqq n \end{pmatrix}, \quad \begin{pmatrix} 0 \leqq h \leqq m \\ n < k < 3n \end{pmatrix}, \quad \begin{pmatrix} 0 \leqq h \leqq m \\ 3n \leqq k \leqq 4n \end{pmatrix},$$
$$\begin{pmatrix} m < h < 3m \\ 0 \leqq k \leqq n \end{pmatrix}, \quad \begin{pmatrix} m < h < 3m \\ n < k < 3n \end{pmatrix}, \quad \begin{pmatrix} m < h < 3m \\ 3n \leqq k \leqq 4n \end{pmatrix},$$
$$\begin{pmatrix} 3m \leqq h \leqq 4m \\ 0 \leqq k \leqq n \end{pmatrix}, \quad \begin{pmatrix} 3m \leqq h \leqq 4m \\ n < k < 3n \end{pmatrix}, \quad \begin{pmatrix} 3m \leqq h \leqq 4m \\ 3n \leqq k \leqq 4n \end{pmatrix}.$$

Since

$$2^{-4m} \sum_{h=0}^{4m} \binom{4m}{h} = 1,$$

$$\lim_{m\to\infty} 2^{-4m} \sum_{h=0}^{m} \binom{4m}{h} = 0 \text{ [2, p. 511]}, \qquad \lim_{m\to\infty} 2^{-4m} \sum_{h=3m}^{4m} \binom{4m}{h} = 0,$$

and A_{mn} is bounded it follows that each of the parts $S_1, S_2, S_3, S_4, S_6, S_7, S_8, S_9$ has the limit zero as $m, n \to \infty$. Thus if S_5 has the limit zero it will follow that the difference $\overset{1}{B}_{4m,4n} - A_{2m,2n}$ has the limit zero.

Let $Q_{m,n}$ denote the largest of the numbers $((m+h)^{1/2} + (n+k)^{1/2}) \cdot ((m+h)(n+k))^{1/2} \cdot |a_{m+h,n+k}|$ for $m < h < 3m$ and $n < k < 3n$. Then for all h, k in these intervals

$$|A_{hk} - A_{2m,2n}|$$
$$\leqq (|2m - h| \cdot 3n + |2n - k| \cdot 2m) \frac{Q_{mn}}{(m^{1/2} + n^{1/2})(mn)^{1/2}}$$

if $mn\neq 0$. Hence

$$|S_5|\leqq 2^{-4m-4n}\sum_{h,k=m+1,n+1}^{3m-1,3n-1}\binom{4m}{h}\binom{4n}{k}(|2m-h|\cdot 3n + |2n-k|\cdot 2m)\frac{Q_{mn}}{(m^{1/2}+n^{1/2})\cdot(mn)^{1/2}}$$

$$\leqq\left\{3n\cdot 2^{-4m}\sum_{h=m+1}^{3m-1}|2m-h|\cdot\binom{4m}{h} + 2m\cdot 2^{-4n}\sum_{k=n+1}^{3n-1}|2n-k|\cdot\binom{4n}{k}\right\}\frac{Q_{mn}}{(m^{1/2}+n^{1/2})\cdot(mn)^{1/2}}.$$

But

$$\sum_{h=m+1}^{3m-1}|2m-h|\cdot\binom{4m}{h}<2\sum_{h=0}^{2m}(2m-h)\binom{4m}{h}$$

and

$$\sum_{h=0}^{2m}(2m-h)\binom{4m}{h}$$

$$=2m\left\{\frac{1}{2}\sum_{h=0}^{4m}\binom{4m}{h}+\frac{1}{2}\binom{4m}{2m}\right\}-4m\sum_{h=1}^{2m}\binom{4m-1}{h-1}$$

$$=m\left\{2^{4m}+\binom{4m}{2m}-4\sum_{h=0}^{2m-1}\binom{4m-1}{h}\right\}=m\binom{4m}{2m}.$$

Hence

$$|S_5|<\left\{6mn\cdot 2^{-4m}\binom{4m}{2m}+4mn\cdot 2^{-4n}\binom{4n}{2n}\right\}\frac{Q_{mn}}{(m^{1/2}+n^{1/2})\cdot(mn)^{1/2}}.$$

Since

$$2^{-2n}\binom{2n}{n}\cong(\pi n)^{-1/2}$$

[2, p. 385] it then follows that

$$|S_5|<\{6mn(2\pi m)^{-1/2}(1+e_m) + 4mn(2\pi n)^{-1/2}(1+e_n)\}\frac{Q_{mn}}{(m^{1/2}+n^{1/2})\cdot(mn)^{1/2}}$$

where $e_m\to 0$ as $m\to\infty$ and $e_n\to 0$ as $n\to\infty$. Thus

$$|S_5| < \left\{\frac{6n^{1/2}(1+e_m)+4m^{1/2}(1+e_n)}{m^{1/2}+n^{1/2}}\right\}\cdot Q_{mn}.$$

Since the quantity in braces is bounded and $Q_{mn}\to 0$ it then follows that $S_5\to 0$ as $m, n\to\infty$. Hence the difference $B^1_{4m,4n}-A_{2m,2n}$ has the limit zero. With only slight modifications of this argument it can be shown that $B^1_{4m,4n}-A_{2m+1,2n}$, $B^1_{4m,4n}-A_{2m,2n+1}$, and $B^1_{4m,4n}-A_{2m+1,2n+1}$ have the limit zero. The proof of the theorem is then complete.

References

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