ON EULER METHODS OF SUMMABILITY FOR DOUBLE SERIES

G. N. WOLLAN

The two qth order Euler transforms of the sequence A_n

$$A_n^q = (q+1)^{-n-1} \sum_{k=0}^{n} \binom{n+1}{k+1} q^{n-k} A_k$$

and

$$B_n^q = (q+1)^{-n} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} A_k$$

are equivalent for $q \geqq 0$ in the sense that if either has a limit as $n \to \infty$ the other has the same limit [1, p. 180].[1] For double sequences the corresponding transforms are

$$A_{mn}^q = (q+1)^{-m-n-2} \sum_{h,k=0}^{m,n} \binom{m+1}{h+1} \binom{n+1}{k+1} q^{m+n-h-k} A_{hk}, \tag{1}$$

$$B_{mn}^q = (q+1)^{-m-n} \sum_{h,k=0}^{m,n} \binom{m}{h} \binom{n}{k} q^{m+n-h-k} A_{hk}. \tag{2}$$

This paper is concerned with two theorems regarding these transforms. Throughout the discussion $q \geqq 0$.

THEOREM 1. *If A_{mn}^q has a limit as $m, n \to \infty$, then B_{mn}^q has that same limit and if B_{mn}^q has a limit and is bounded, then A_{mn}^q has that same limit but there do exist sequences for which B_{mn}^q has a limit but for which* $\lim_{m,n\to\infty} A_{mn}^q$ *does not exist for any $q \geqq 0$.*

The relation

$$B_{mn}^q = q^2 A_{m-1,n-1}^q - q(q+1)(A_{m,n-1}^q + A_{m-1,n}^q) + (q+1)^2 A_{mn}^q \tag{3}$$

may be verified by substitution from (1) into the right-hand side. This relation may be written in the form

$$\begin{aligned} B_{mn}^q = {} & q^2(A_{m-1,n-1}^q - A_{m,n-1}^q - A_{m-1,n}^q + A_{mn}^q) \\ & - q(A_{m,n-1}^q + A_{m-1,n}^q - 2A_{mn}^q) + A_{mn}^q. \end{aligned} \tag{4}$$

Presented to the Society, November 29, 1952; received by the editors December 4, 1952.

[1] Numbers in brackets refer to the references at the end of the paper.

From this relation (4) it follows that if $\lim_{m,n\to\infty} A^q_{mn}=A$, then $\lim_{m,n\to\infty} B^q_{mn}=A$.

Relation (3) can be used to express A^q_{mn} in terms of B^q_{mn}. First write (3) in the form

$$(q+1)^2 A^q_{mn} = B^q_{mn} + q(q+1)(A^q_{m,n-1} + A^q_{m-1,n}) - q^2 A^q_{m-1,n-1}.$$

In this replace $A^q_{m,n-1}$ and $A^q_{m-1,n}$, by the values which this relation gives for them. This yields

$$\begin{aligned}(q+1)^2 A^q_{mn} = B^q_{mn} &+ q(q+1)^{-1}(B^q_{m,n-1} + B^q_{m-1,n}) \\ &+ q^2(A^q_{m,n-2} + A^q_{m-1,n-1} + A^q_{m-2,n}) \\ &- q^3(q+1)^{-1}(A^q_{m-1,n-2} + A^q_{m-2,n-1}).\end{aligned}$$

Successive repetitions of this procedure lead finally to the relation

$$(q+1)^2 A^q_{mn} = \sum_{h,k=0}^{m,n} \left(\frac{q}{q+1}\right)^{m+n-h-k} B^q_{hk}. \tag{5}$$

Relation (5) expresses A^q_{mn} as a transform of the sequence B^q_{mn}. The coefficients of the transformation satisfy the conditions for regularity [3, p. 23]. Hence if B^q_{mn} has the limit A and is bounded, then A^q_{mn} also has the limit A.

To see that there exist sequences for which the transform B^q_{mn} has a limit but for which $\lim_{m,n\to\infty} A^q_{mn}$ does not exist for any $q\geqq 0$ consider the sequence $A_{mn}=(-1)^n p^{2m+n-1}\{n(p+1)+p\}$, $p>1$. For this sequence one may readily verify by substitution into (2) that $B^p_{mn}=0$ whenever $n>1$. Thus for this sequence the transform B^p_{mn} has the limit 0. But by substituting into (1) and simplifying one obtains

$$\begin{aligned}A^q_{mn} = &\left\{\left(\frac{q+p^2}{q+1}\right)^{m+1} - \left(\frac{q}{q+1}\right)^{m+1}\right\} \\ &\cdot\left(\frac{n+1}{q+1}\right)\cdot\left(\frac{p+1}{p^3}\right)\cdot\left(\frac{q-p}{q+1}\right)^n \\ &+ p^{-3}\left\{\left(\frac{q+p^2}{q+1}\right)^{m+1} - \left(\frac{q}{q+1}\right)^{m+1}\right\} \\ &\left\{\left(\frac{q-p}{q+1}\right)^{n+1} - \left(\frac{q}{q+1}\right)^{n+1}\right\}\end{aligned}$$

and for $p>1$ this does not have a limit for any $q\geqq 0$. This completes the proof of Theorem 1.

THEOREM 2. *Let* $A_{mn}=\sum_{h,k=0}^{m,n} a_{hk}$. If

$$(1)\ B^1_{mn} = 2^{-m-n}\sum_{h,k=0}^{m,n}\binom{m}{h}\binom{n}{k}A_{hk}$$

has the limit A *as* $m,\ n\to\infty$, (2) A_{mn} *is bounded and*

$$(3)\quad \lim_{m,n\to\infty}(m^{1/2}+n^{1/2})(mn)^{1/2}a_{mn}=0,$$

then A_{mn} *also has the limit* A *as* $m,\ n\to\infty$.

Form the difference

$$B^1_{4m,4n}-A_{2m,2n}=2^{-4m-4n}\sum_{h,k=0}^{4m,4n}\binom{4m}{h}\binom{4n}{k}(A_{hk}-A_{2m,2n}).$$

Separate this difference into 9 parts $S_1,\ S_2,\ \cdots,\ S_9$ corresponding respectively to the intervals of summation

$$\begin{pmatrix}0\leqq h\leqq m\\ 0\leqq k\leqq n\end{pmatrix},\quad \begin{pmatrix}0\leqq h\leqq m\\ n<k<3n\end{pmatrix},\quad \begin{pmatrix}0\leqq h\leqq m\\ 3n\leqq k\leqq 4n\end{pmatrix},$$
$$\begin{pmatrix}m<h<3m\\ 0\leqq k\leqq n\end{pmatrix},\quad \begin{pmatrix}m<h<3m\\ n<k<3n\end{pmatrix},\quad \begin{pmatrix}m<h<3m\\ 3n\leqq k\leqq 4n\end{pmatrix},$$
$$\begin{pmatrix}3m\leqq h\leqq 4m\\ 0\leqq k\leqq n\end{pmatrix},\quad \begin{pmatrix}3m\leqq h\leqq 4m\\ n<k<3n\end{pmatrix},\quad \begin{pmatrix}3m\leqq h\leqq 4m\\ 3n\leqq k\leqq 4n\end{pmatrix}.$$

Since

$$2^{-4m}\sum_{h=0}^{4m}\binom{4m}{h}=1,$$

$$\lim_{m\to\infty}2^{-4m}\sum_{h=0}^{m}\binom{4m}{h}=0\ [\mathbf{2},\ \text{p. } 511],\qquad \lim_{m\to\infty}2^{-4m}\sum_{h=3m}^{4m}\binom{4m}{h}=0,$$

and A_{mn} is bounded it follows that each of the parts $S_1,\ S_2,\ S_3,\ S_4,\ S_6,\ S_7,\ S_8,\ S_9$ has the limit zero as $m,\ n\to\infty$. Thus if S_5 has the limit zero it will follow that the difference $B^1_{4m,4n}-A_{2m,2n}$ has the limit zero.

Let $Q_{m,n}$ denote the largest of the numbers $((m+h)^{1/2}+(n+k)^{1/2})\cdot((m+h)(n+k))^{1/2}\cdot|a_{m+h,n+k}|$ for $m<h<3m$ and $n<k<3n$. Then for all $h,\ k$ in these intervals

$$|A_{hk}-A_{2m,2n}|$$
$$\leqq(|2m-h|\cdot 3n+|2n-k|\cdot 2m)\frac{Q_{mn}}{(m^{1/2}+n^{1/2})(mn)^{1/2}}$$

if $mn \neq 0$. Hence

$$|S_5| \leqq 2^{-4m-4n} \sum_{h,k=m+1,n+1}^{3m-1,3n-1} \binom{4m}{h}\binom{4n}{k}(|2m-h|\cdot 3n + |2n-k|\cdot 2m)\frac{Q_{mn}}{(m^{1/2}+n^{1/2})\cdot(mn)^{1/2}}$$

$$\leqq \left\{3n\cdot 2^{-4m}\sum_{h=m+1}^{3m-1}|2m-h|\cdot\binom{4m}{h} + 2m\cdot 2^{-4n}\sum_{k=n+1}^{3n-1}|2n-k|\cdot\binom{4n}{k}\right\}\frac{Q_{mn}}{(m^{1/2}+n^{1/2})\cdot(mn)^{1/2}}.$$

But

$$\sum_{h=m+1}^{3m-1}|2m-h|\cdot\binom{4m}{h} < 2\sum_{h=0}^{2m}(2m-h)\binom{4m}{h}$$

and

$$\sum_{h=0}^{2m}(2m-h)\binom{4m}{h} = 2m\left\{\frac{1}{2}\sum_{h=0}^{4m}\binom{4m}{h}+\frac{1}{2}\binom{4m}{2m}\right\} - 4m\sum_{h=1}^{2m}\binom{4m-1}{h-1}$$

$$= m\left\{2^{4m}+\binom{4m}{2m} - 4\sum_{h=0}^{2m-1}\binom{4m-1}{h}\right\} = m\binom{4m}{2m}.$$

Hence

$$|S_5| < \left\{6mn\cdot 2^{-4m}\binom{4m}{2m} + 4mn\cdot 2^{-4n}\binom{4n}{2n}\right\}\frac{Q_{mn}}{(m^{1/2}+n^{1/2})\cdot(mn)^{1/2}}.$$

Since

$$2^{-2n}\binom{2n}{n} \cong (\pi n)^{-1/2}$$

[2, p. 385] it then follows that

$$|S_5| < \{6mn(2\pi m)^{-1/2}(1+e_m) + 4mn(2\pi n)^{-1/2}(1+e_n)\}\frac{Q_{mn}}{(m^{1/2}+n^{1/2})\cdot(mn)^{1/2}}$$

where $e_m \to 0$ as $m \to \infty$ and $e_n \to 0$ as $n \to \infty$. Thus

$$|S_5| < \left\{\frac{6n^{1/2}(1 + e_m) + 4m^{1/2}(1 + e_n)}{m^{1/2} + n^{1/2}}\right\} \cdot Q_{mn}.$$

Since the quantity in braces is bounded and $Q_{mn} \to 0$ it then follows that $S_5 \to 0$ as $m, n \to \infty$. Hence the difference $B^1_{4m,4n} - A_{2m,2n}$ has the limit zero. With only slight modifications of this argument it can be shown that $B^1_{4m,4n} - A_{2m+1,2n}$, $B^1_{4m,4n} - A_{2m,2n+1}$, and $B^1_{4m,4n} - A_{2m+1,2n+1}$ have the limit zero. The proof of the theorem is then complete.

References

1. G. H. Hardy, *Divergent series*, Oxford, 1949.
2. K. Knopp, *Theory and applications of infinite series*, London, 1928.
3. C. N. Moore, *Summable series and convergence factors*, Amer. Math. Soc. Colloquium Publications, vol. 22, 1938.

Memphis State College