A NONCONVERGENT ITERATIVE PROCESS

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1. **Introduction.** In [1] Mann and Wolf considered the integral equation

$$(1) \qquad y(t) = \int_0^t \frac{G[y(x)]}{[\pi(t-x)]^{1/2}}\,dx,$$

where

(2) $G(y)$ is continuous and strictly decreasing for positive y, and

$$G(1) = 0.$$

They defined a sequence of functions $y_0(t)$, $y_1(t)$, $\cdots$ inductively as follows:

$$(3) \qquad y_0(t) = 0, \qquad y_{n+1}(t) = \int_0^t \frac{G[y_n^*(x)]}{[\pi(t-x)]^{1/2}}\,dx,$$

where $y_n^*(x) = \min\,(y_n(x), 1)$. Under the additional assumption that $G(y)$ satisfies a Lipschitz condition on [0, 1] they proved that the sequence $y_0(t)$, $y_1(t)$, $\cdots$ converges to a bounded solution,[1] $y(t)$, of (1). Dr. Mann pointed out to me that it was not known whether or not the requirement of a Lipschitz condition was superfluous. The present paper resolves this uncertainty by giving an example of a function $G(y)$ satisfying (2) for which the corresponding sequence (3) does not converge. It also contains a positive result, to the effect that the sequence defined by (3) does converge to the solution $y(t)$ if, in addition to requirement (2), $G(y)$ is convex.

2. **The counter example.** The desired function $G(y)$ is defined as follows:

$$(4) \qquad \begin{aligned} G(y) &= 1 - y && \text{for } 0 \leqq y \leqq 1/2; \\ G(y) &= [1 - (2y-1)^{1/3}]/2 && \text{for } 1/2 < y. \end{aligned}$$

Let $G_1(y) = 1 - y$ for $y \geqq 0$, and let $z(t)$ be the bounded solution of

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[1] It was shown in [2] that even in the absence of the Lipschitz condition equation (1) has a unique bounded solution $y(t)$, provided only that G satisfies requirement (2). This solution $y(t)$ is strictly increasing and approaches the limit 1 as t increases indefinitely.

$$z(t) = \int_0^t \frac{G_1[z(x)]}{[\pi(t-x)]^{1/2}}\,dx. \tag{5}$$

Now, as was shown in [1, p. 168], $z(t)$ is continuous ($t \geqq 0$) and dz/dt is positive and decreasing for $t>0$. Thus a positive number α is uniquely determined by the requirement $z(\alpha)=1/2$. Let k be dz/dt evaluated at 2α, and let β be the smaller of 2α and $\alpha+4\alpha^3k^2$. Then clearly

$$z(t) \geqq 1/2 + k(t-\alpha) \qquad (\alpha \leqq t \leqq \beta). \tag{6}$$

Lemma 1. *If, for some n, $y_n(t) \geqq z(t)$ for $0 \leqq t \leqq \beta$, then over the same interval $y_{n+1}(t) \leqq \min\,(z(t),\,1/2)$.*

Lemma 2. *If, for some n, $y_n(t) \leqq \min\,(z(t),\,1/2)$ for $0 \leqq t \leqq \beta$, then over the same interval $y_{n+1}(t) \geqq z(t)$.*

Assuming that these lemmas are true, then $y_{2r}(t) \leqq \min\,(z(t),\,1/2)$, and $y_{2r+1}(t) \geqq z(t)$, on $0 \leqq t \leqq \beta$. Then clearly the sequence $y_0(t)$, $y_1(t), \cdots$ does not converge for any t between α and β because $z(t) > 1/2$ over this range.

Proof of Lemma 1. Define $Y(t)$ as follows:

$$Y(t) = \int_0^t \frac{G[z(x)]}{[\pi(t-x)]^{1/2}}\,dx. \tag{7}$$

Since $y_n(x) \geqq z(x)$ for $0 \leqq x \leqq \beta$ and G is a decreasing function, we have $y_{n+1}(t) \leqq Y(t)$. Now $Y(t)=z(t)$ for $0 \leqq t \leqq \alpha$; we shall show that $Y(t) < 1/2$ for $\alpha < t \leqq \beta$. Throughout the remainder of the proof t will be a fixed number $\alpha+\Delta t$, $0<\Delta t \leqq \beta-\alpha$. From (7) we have

$$\begin{aligned} Y(t) - Y(\alpha) &= \int_\alpha^{\alpha+\Delta t} \frac{G[z(x)]}{[\pi(\alpha+\Delta t-x)]^{1/2}}\,dx \\ &\quad - \int_0^\alpha G[z(x)]\left[\frac{1}{[\pi(\alpha-x)]^{1/2}} - \frac{1}{[\pi(\alpha+\Delta t-x)]^{1/2}}\right]dx, \\ &= \text{Gain} - \text{Loss, say.} \end{aligned} \tag{8}$$

Now $G[z(x)] \geqq 1/2$ over $0 \leqq x \leqq \alpha$, and integration gives

$$\text{Loss} \geqq \pi^{-1/2}(\alpha^{1/2} - (\alpha+\Delta t)^{1/2} + (\Delta t)^{1/2}). \tag{9}$$

To get an upper bound on the gain in (8) we first use (6) and (4), and find that $G[z(x)] \leqq (1-[2k(x-\alpha)]^{1/3})/2 = f(x)$, say. Next we replace $f(x)$ by the linear function $F(x)$ determined to equal $f(x)$ at $x=\alpha$ and at $x=\alpha+\Delta t$. Since $f(x)$ is convex we clearly have $f(x) \leqq F(x)$ $(\alpha \leqq x \leqq \alpha+\Delta t)$, and thus

$$(10) \quad G[z(x)] \leqq F(x) = 1/2 - m(x-\alpha), \text{ where } m = (\Delta t)^{-2/3}(2k)^{1/3}/2.$$

Substituting for $G[z(x)]$ in the first integral of (8) and performing the integration gives

$$(11) \qquad \text{Gain} \leqq \pi^{-1/2}[(\Delta t)^{1/2} - (4m/3)(\Delta t)^{3/2}].$$

Then, from (11) and (9),

$$(12) \qquad \begin{aligned} \text{Gain} - \text{Loss} &\leqq \pi^{-1/2}((\alpha+\Delta t)^{1/2} - \alpha^{1/2} - (4m/3)(\Delta t)^{3/2}) \\ &\leqq \pi^{-1/2}[\Delta t(\alpha^{-1/2}/2) - (4m/3)(\Delta t)^{3/2}] \\ &= (\pi^{-1/2}\Delta t)[(\alpha^{-1/2}/2) - (4m/3)(\Delta t)^{1/2}]. \end{aligned}$$

In the last member of (12) replace m by its value (see (10)), and replace Δt (in the second factor) by its upper bound, $4\alpha^3 k^2$. This gives

$$(13) \qquad \text{Gain} - \text{Loss} \leqq \pi^{-1/2}\Delta t[\alpha^{-1/2}/2 - 2\alpha^{-1/2}/3] < 0.$$

This completes the proof of Lemma 1.

Proof of Lemma 2. Now $G(y) = G_1(y)$ for $0 \leqq y \leqq 1/2$, so under the hypothesis that $y_n(t) \leqq \min(z(t), 1/2)$ we know that

$$G[y_n(x)] = G_1[y_n(x)] \qquad \text{for } 0 \leqq x \leqq \beta.$$

Then over this range

$$y_{n+1}(t) = \int_0^t \frac{G_1[y_n(x)]}{[\pi(t-x)]^{1/2}}\,dx \geqq \int_0^t \frac{G_1[z(x)]}{[\pi(t-x)]^{1/2}}\,dx = z(t).$$

With this proof of Lemma 2 our discussion of the counterexample is complete.

3. The theorem. *If $G(y)$ satisfies* (2) *and in addition is convex for $0 \leqq y \leqq 1$, then the sequence $y_0(t), y_1(t), \cdots$ given by* (3) *converges to the solution $y(t)$ of* (1).

Proof. Now $y_0(t)=0$ and $y_1(t) = 2G(0)(t/\pi)^{1/2}$. Define positive numbers d and c by the respective requirements

$$(14) \qquad G(d) = 3G(0)/4, \qquad y_1(c) = d.$$

We first prove the conclusion of the theorem for t restricted to the interval $[0, c]$.

From the convexity of $G(y)$ we see that for any r_1 and r_2 between 0 and 1 ($r_1 \neq r_2$) we have

$$\left|\frac{G(r_1) - G(r_2)}{r_1 - r_2}\right| \leqq \frac{G(0) - G(r_1)}{r_1}. \tag{15}$$

From our choice of d and c and the fact that $y_n(t) \leqq y_1(t)$ for all n we see from (2) that $G[y_n(t)] \geqq 3G(0)/4$ for $0 \leqq t \leqq c$. Then

$$y_n(t) \geqq (3/4)y_1(t) = (3/2)G(0)(t/\pi)^{1/2}. \tag{16}$$

Let $\Delta_n = \max|y_n(t) - y_{n-1}(t)|$ for $0 \leqq t \leqq c$. Thus

$$\begin{aligned} |y_{n+1}(t) - y_n(t)| &= \int_0^t \frac{|G[y_n(x)] - G[y_{n-1}(x)]|}{[\pi(t-x)]^{1/2}} dx \\ &\leqq \int_0^t \frac{(G(0) - G[y_n(x)]) \cdot |y_n(x) - y_{n-1}(x)|}{y_n(x)[\pi(t-x)]^{1/2}} dx \\ &\leqq \int_0^t \frac{[G(0)/4]\Delta_n}{(3/2)G(0)(x/\pi)^{1/2}[\pi(t-x)]^{1/2}} dx \\ &= \frac{\Delta_n}{6}\int_0^t \frac{dx}{[x(t-x)]^{1/2}} = (\pi/6)\Delta_n. \end{aligned}$$

(In the above we first use (3), then (15), and then (14) and (16). The two final equalities are obvious.) Thus $|y_{n+1}(t) - y_n(t)| \leqq \Delta_{n+1} \leqq \Delta_n(\pi/6) \leqq (\pi/6)^n$, for $0 \leqq t \leqq c$. This proves the convergence on the interval $[0, c]$.

Suppose the theorem is false and that on some interval $[0, T]$ the sequence $y_0(t), y_1(t), \cdots$ does not converge. Now for every t, $y_0(t) \leqq y_2(t) \leqq \cdots \leqq y(t) \leqq \cdots \leqq y_3(t) \leqq y_1(t)$. Therefore the y's of even subscript converge to a continuous limit function $Y_1(t)$ and the y's of odd subscript converge to a continuous limit function $Y_2(t)$, and $Y_1(t) \leqq y(t) \leqq Y_2(t)$. It is furthermore clear that the substitution of $Y_1(x)$ [respectively $Y_2(x)$] for $y(x)$ under the integral sign in (1) gives $Y_2(t)$ [respectively $Y_1(t)$] in place of $y(t)$. The convergence of $y_0(t)$, $y_1(t), \cdots$ on $[0, c]$ implies that $Y_1(t) = Y_2(t)$ for $0 \leqq t \leqq c$. Let e be the greatest number such that $Y_1(t) = Y_2(t)$ for $0 \leqq t \leqq e$. Then $c \leqq e < T$.

Since Y_1 has a positive minimum value on $[e, T]$ it follows from the hypotheses on G that there exists a positive k such that for any x on $[e, T]$, $G[Y_1(x)] - G[Y_2(x)] \leqq k|Y_1(x) - Y_2(x)|$. Choose a fixed t ($e < t < T$) so that (a) $2k[(t-e)/\pi]^{1/2} < 1$ and (b) $|Y_1(x) - Y_2(x)| \leqq |Y_1(t) - Y_2(t)|$ for $e \leqq x \leqq t$. Then

$$
\begin{aligned}
| Y_2(t) - Y_1(t) | &= \int_0^t \frac{G[Y_1(x)] - G[Y_2(x)]}{[\pi(t-x)]^{1/2}}\,dx \\
&\leqq \int_e^t \frac{k\,| Y_1(x) - Y_2(x) |}{[\pi(t-x)]^{1/2}}\,dx \\
&\leqq k\,| Y_1(t) - Y_2(t) | \int_e^t \frac{dx}{[\pi(t-x)]^{1/2}} \\
&= 2k[(t-e)/\pi]^{1/2} \cdot | Y_1(t) - Y_2(t) | \\
&< Y_2(t) - Y_1(t).
\end{aligned}
$$

Thus the assumption that the theorem is false has led to a contradiction.

References

1. W. Robert Mann and František Wolf, *Heat transfer between solids and gasses under nonlinear boundary conditions*, Quarterly of Applied Mathematics vol. 9 (1951) pp. 163–184.

2. J. H. Roberts and W. R. Mann, *On a certain nonlinear integral equation of Volterra type*, Pacific Journal of Mathematics vol. 1 (1951) pp. 431–445.

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