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DENSE IMBEDDING OF TOPOLOGICAL GROUPS

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The present article originated from the problem of determining when a continuous representation of a Lie group into another is open. Important cases of the problem have been discussed by A. Malcev[2] and the author,[3] and some of them have been extended to the case of more general groups by the author and H. Yamabe.[4]

Recently W. T. van Est obtained new results concerning the same problem on Lie groups.[5] Here I shall give an extension of his essential result to a more general case in a simpler way.

Let G be a locally compact connected group and let $A(G)$ be the group of all continuous automorphisms of G. Let $A(G)$ be topologized by the notion of uniform convergence in the wider sense.

Now let $I(G)$ be the subgroup of $A(G)$ composed of all inner automorphisms of G. We shall call G a (CA) group[6] if $I(G)$ is a closed subgroup of $A(G)$.

Lemma.[7] *Let G be a locally compact connected and locally connected group, and H a locally compact group. If ϕ is a continuous isomorphism which maps G in an everywhere dense subgroup in H, then the following propositions hold:*

(1) *$\phi(G)$ is an invariant subgroup of H.*

(2) *Let h be an element of H. Let us consider the automorphism $\sigma_h(x)$ defined by $\sigma_h x = \phi^{-1}(h^{-1}\phi(x))h$ for $x \in G$. Then $\sigma_h(x)$ is a continuous*

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[1] The author's name will be hereafter spelled "Goto" instead of "Gotô" which has been used thus far.

[2] Malcev [4] in the bibliography.

[3] Goto [1].

[4] Goto and Yamabe [3]. See also [2].

[5] van Est [5], where he solved a prize problem (Wiskundig Genootshap Amsterdam, 1950), which had already been established in [1] and generalized in [3], independently of the author.

[6] The notion of a (CA) group is a generalization of van Est's (CA) Lie group.

[7] See [3] and [2].

automorphism of G.

(3) *The correspondence* $h \to \sigma_h$ *defines a continuous homomorphism from H into* $A(G)$.

From this lemma we infer the following theorem.

THEOREM.[8] *Let G be a locally compact connected and locally connected* (CA) *group, and H a locally compact group. Suppose that the center Z of G is compact. If there exists a continuous isomorphism* ϕ *mapping G into H, then the image* $\phi(G)$ *is a closed subgroup of H.*

PROOF. Clearly we need consider only the case when H coincides with the closure $\mathrm{Cl}(\phi(G))$ of $\phi(G)$. In such a case we get a continuous homomorphism $h \to \sigma_h$ from H into $A(G)$ by virtue of the above lemma, and $I(G)$ corresponds to $\phi(G)$ under this homomorphism: $\sigma\phi(G) = I(G)$. Now since σ is continuous we get $\sigma\mathrm{Cl}(\phi(G)) \subseteq \mathrm{Cl}(I(G))$ and thus $\sigma(H) \subseteq I(G)$.[9] The above relation means that for every h in H there exists an element g of G such that

$$\phi(g)^{-1}x\phi(g) = h^{-1}xh \qquad \text{for all } x \in \phi(G).$$

Then $h\phi(g)^{-1}$ commutes with every element of $\phi(G)$, whence $h\phi(g)^{-1}$ is contained in the center A of H. Thus we have

$$H = \phi(G)A.$$

Since Z is compact, $\phi(Z)$ is also compact, and the relation $\phi(G) \cap A = \phi(Z)$ is obvious. Hence we have algebraically

$$H/\phi(Z) = \phi(G)/\phi(Z) \times A/\phi(Z)$$

where $\times$ means the direct product of abstract groups.

Let us now consider the topological direct product group

$$L = G/Z \times A/\phi(Z)$$

of locally compact groups G/Z and $A/\phi(Z)$. It is easy to show that L can be covered by countable compact sets. Hence the continuous isomorphism from L onto $H/\phi(Z)$, obtained by extending the mapping ϕ and the identity mapping of $A/\phi(Z)$, is necessarily open. Therefore $\phi(G)/\phi(Z)$ is closed in $H/\phi(Z)$, whence $\phi(G)$ is closed in H; this completes the proof.

COROLLARY.[10] *Let* $\mathcal{G}$ *be a local Lie group. Then* $\mathcal{G}$ *always generates a*

[8] This theorem includes the essential theorems in both [3] and [5].

[9] The following argument is entirely the same as one in [3].

[10] Our theorem is superfluous for obtaining this corollary, which can be proved by the theorem of van Est only.

closed subgroup whenever $\mathcal{G}$ is imbedded in a Lie group as a local subgroup if and only if the following two conditions are satisfied:

(1) *The adjoint group of $\mathcal{G}$ is a closed linear group.*

(2) *The center of the simply-connected group corresponding to $\mathcal{G}$ is a finite group.*

PROOF. Necessity: (2) is already known.[11] Then as the adjoint group of $\mathcal{G}$ is locally isomorphic with $\mathcal{G}$, it must be closed in the general linear group.

The sufficiency follows readily from the theorem.

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[11] See [1] and [5].