LOCAL CROSS SECTIONS IN LOCALLY COMPACT GROUPS

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Introduction. Let p be the natural projection of the topological group G, with subgroup H, onto the coset space G/H. The subgroup H is said to have a *local cross section* if there exists an open set U in G/H, and a continuous function f defined on U with values in G such that $pf(x)=x$ for x in U. The most general conditions on G and H under which such a function exists are not known. It has been conjectured [**7**, p. 33][2] that if G is compact and of finite dimension, then H has a local cross section. (For the infinite-dimensional case, there are examples of compact groups with closed subgroups not having a local cross section.) In this paper, we show that if G is locally compact, separable, metric, and of finite dimension, and H is a closed subgroup of G, then H has a local cross section. In §1, several elementary lemmas necessary for the proof are stated, along with certain properties of Lie groups and projective limits. In §2, we prove the main theorem.

1. **Preliminary definitions and theorems.** We shall, in the following, use, principally, the notation and terminology of [**4**]. Let $\{G_k\}$ be a sequence of groups indexed by the positive integers. Suppose there exists, for each k, a continuous, open homomorphism π_k^{k+1} of G_{k+1} onto G_k. Let $G^*=P_kG_k$. Then the group $G=[x=\{x_k\}\mid \pi_k^{k+1}(x_{k+1})=x_k]$ is called the *projective limit* of the sequence $\{G_k\}$ and π_k is the *projection* of G on G_k. The following properties of projective limit groups may be found in [**4**, pp. 54–56], [**2**, pp. 212–232], or else are easily verified.

LEMMA 1. *If H is a closed subgroup of G, where G is the projective limit of Lie groups, and H_k is the natural projection $\pi_k(H)$ in G_k, then H is the projective limit of $\{H_k\}$, and G/H is the projective limit of $\{G_k/H_k\}$.*

G^* has the usual Tychonoff topology. G is a closed subgroup of G^* under the induced topology. An open subset in G, then, contains an

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[2] Numbers in brackets refer to the references at the end of the paper.

open set of the form $G\cap P_k W_k$ where

$$W_k = \begin{cases} V_k \text{ open in } G_k, & k = k_1, \cdots, k_n, \\ G_k \text{ otherwise.} \end{cases}$$

Let $(\pi_k^{k+1})^{-1} = \pi_{k+1}^k$, $\pi_k^{k+1} \cdots \pi_{k+m-1}^{k+m} = \pi_k^{k+m}$, π_k^k = identity map in G_k. Define

$$U_k = \bigcap_{m=k_1}^{k_n} \pi_k^m (W_m).$$

Then $G\cap P_k U_k = G\cap P_k W_k$. Similar open sets are in G/H.

THEOREM 1. *Let G be the projective limit of the sequence $\{G_k\}$, H a closed subgroup of G, $H_k = \pi_k(H)$, and $\bar{\pi}_k^{k+1}$ the induced mapping $G_{k+1}/H_{k+1} \to G_k/H_k$. Then a sufficient condition for H to have a local cross section is that for some open set $U = (G/H)\cap P_k U_k$, as defined above, there exist local cross sections f_k: $U_k \to G_k$ such that $\pi_k^{k+1} f_{k+1} = f_k \bar{\pi}_k^{k+1}$.*

PROOF. Define $f(\{x_k\}) = \{f_k(x_k)\}$. One may easily verify that f satisfies the conditions of a local cross section.

LEMMA 2. *Let G_1, G_2 be two Lie groups (of the same dimension), π a continuous, open homomorphism, with finite kernel, of G_2 onto G_1. If V_1 is an open n-cell neighborhood of the identity e in G_1, then there exists a set V_2^* homeomorphic, under π, with V_1, and if $\pi^{-1}(e) = [s_1, \cdots, s_m]$,*

$$V_2 = \pi^{-1}(V_1) = \bigcup_{i=1}^{m} s_i V_2^*.$$

LEMMA 3. *Under the hypothesis of Lemma 2, if H_2 is a closed subgroup of G_2, $H_1 = \pi(H_2)$, $U_1 = p_1(V_1)$, $U_2^* = p_2(V_2^*)$, where p_i is the natural projection $G_i \to G_i/H_i$, then U_1 is homeomorphic with U_2^*.*

Lemmas 2 and 3 are straightforward and easy in proof.

2. **Principal results.** In this section, all groups are assumed to be separable, metric groups.

THEOREM 2. *If H is a closed subgroup of a 0-dimensional compact group G, then H has a cross section.*

A *cross section* of H is a local cross section for which $U = G/H$. From [7, pp. 31 and 36], we have the following consequence:

COROLLARY. *Under the hypothesis of Theorem 2, G is H equivalent to the product bundle $H\times G/H$.*

PROOF OF THEOREM 2. G is the projective limit of the sequence

$\{G_k\}$ of finite groups. $G_1=[s_1^1, \cdots, s_{n_1}^1]$ where we assume s_i^1, $i=1, \cdots, m_1$, are distinct mod H_1, and are isomorphic to G_1/H_1. (See Theorem 1 for the definition of H_k.) The set $[s_1^1, \cdots, s_{m_1}^1]$ is a cross section f_1 of H_1. $G_2=[s_1^2, \cdots, s_{n_2}^2]$ where we order the elements so that $\pi_1^2(s_i^2)=s_i^1$ for $i=1, \cdots, m_1$, $s_1^2, \cdots, s_{m_2}^2$, $m_2 \geqq m_1$, are distinct mod H_2, and are isomorphic to G_2/H_2. This is possible since $\pi_1^2(H_2)=H_1$. Then the set $[s_1^2, \cdots, s_{m_2}^2]$ is a cross section f_2 of H_2 which agrees (in the sense of Theorem 1) with f_1. Continuing this process, we obtain cross sections satisfying the hypothesis of Theorem 1.

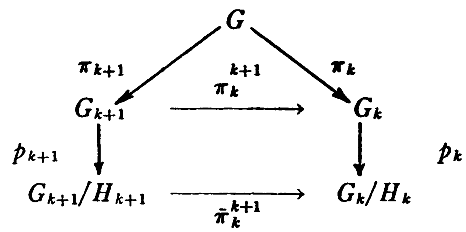
LEMMA 4. *Let G be a locally compact group which is the projective limit of the Lie groups* $\{G_k\}$, *where* $G_k=G/N_k$, N_k *a compact normal subgroup of G. Then* $\pi_{k+1}^k(e)$ *is compact.*

Henceforth, we shall mean projective limit in the sense of Lemma 4 when we say simply "projective limit."

THEOREM 3. *Let G be a locally compact group of finite dimension, H a closed subgroup of G. Then H has a local cross section.*

PROOF. We may assume, because of the results of Gleason [3] and Montgomery-Zippin [5], that G is the projective limit of Lie groups. Also, these groups may be assumed to be of the same dimension (see [6, p. 214]). Hence, $\pi_{k+1}^k(e)$ is finite for every k by Lemma 4. We shall construct local cross sections satisfying the hypothesis of Theorem 1.

Let us keep in mind the following diagram:



There exists an open set U_1' in G_1/H_1 containing H_1, on which a local cross section is defined. (H_1 is closed if H is, so the above statement follows from [1, Proposition 1, p. 110].) Let V_1 be an open n-cell neighborhood of the identity which is contained in $p_1^{-1}(U_1')$. Construct neighborhoods $V_k=\bigcup_{i=1}^{n_k} s_i^k V_k^e$, $k>1$, by Lemmas 2 and 3, where $[s_1^k, \cdots, s_{n_k}^k]=K_k=\pi_k^1(e)$, the s_i^k being ordered as below. Let θ_k be the homeomorphism of V_1 onto V_k^e (see Lemma 2) induced by π_1^k, and $\bar{\theta}_k$ be the homeomorphism, induced by θ_k, of U_1 onto $U_k^e=p_k(V_k^e)$. We order the s_i^2 as follows: s_i^2 are distinct mod H_2 for $i=1, \cdots, m_2 \leqq n_2$, and are isomorphic to the set of left cosets of H_2 in

K_2H_2. Then[3]

$$\bar{\pi}_2^1(U_1) = \bigcup_{i=1}^{m_2} \overset{2}{s}_i U_2^e = \bigcup_{i=1}^{m_2} p_2(\overset{2}{s}_i p_2^{-1}(U_2^e)) = \bigcup_{i=1}^{m_2} p_2(\overset{2}{s}_i V_2^e).$$

Define f_2 on U_2^e by

$$f_2(x) = \theta_2 f_1 \bar{\pi}_1^2(x).$$

Since $s_i^2 U_2^e$ is disjoint from $s_j^2 U_2^e$ if $i \neq j$, $i, j \leqq m_2$, we may define f_2 on $s_i^2 U_2^e$ as

$$f_2(\overset{2}{s}_i x) = \overset{2}{s}_i f_2(x).$$

f_2 is obviously continuous. That it satisfies the conditions of a local cross section is immediate from $p_2 = \bar{\theta}_2 p_1 \pi_1^2$ on U_2^e, and

$$p_2(\overset{2}{s}_i f_2(x)) = \overset{2}{s}_i p_2(f_2(x)) = \overset{2}{s}_i x.$$

Also

$$\pi_1^2 f_2(\overset{2}{s}_i x) = \pi_1^2(\overset{2}{s}_i)\pi_1^2 f_2(x) = \pi_1^2 f_2(x) = \pi_1^2\theta_2 f_1 \bar{\pi}_1^2(x) = f_1\bar{\pi}_1^2(\overset{2}{s}_i x).$$

We now order K_3. Choose s_i^3, $i=1, \cdots, m_2$, so that $\pi_2^3(s_i^3) = s_i^2$ and s_i^3, $i=1, \cdots, m_3$, $n_3 \geqq m_3 \geqq m_2$, are in different left cosets of H_3, and are isomorphic to the set of left cosets of H_3 in K_3H_3. This we can do since $\pi_3^2(K_2) = K_3$. The s_i^3 may be chosen arbitrarily for $i > m_3$. Define the local cross section f_3 on U_3^e by

$$f_3(x) = \theta_3 f_1 \bar{\pi}_1^3(x),$$

and similar to f_2 on $s_i^3 U_3^e$ for $i=1, \cdots, m_3$. Then f_3 is a local cross section on $\bar{\pi}_3^1(U_1) = \bar{\pi}_3^2(U_2)$, where

$$U_2 = \bigcup_{i=1}^{m_2} \overset{2}{s}_i U_2^e,$$

and

$$\begin{aligned} \pi_2^3 f_3(\overset{3}{s}_i x) &= \pi_2^3(\overset{3}{s}_i)\pi_2^3(f_3(x)) = \overset{2}{s}_i \pi_2^3 \theta_3 f_1(\bar{\pi}_1^3(x)) \\ &= \overset{2}{s}_i \theta_2 f_1(\bar{\pi}_1^3(x)) = \overset{2}{s}_i f_2 \bar{\theta}_2 \bar{\pi}_1^3(x) = \overset{2}{s}_i f_2(\bar{\pi}_2^3(x)) \\ &= f_2(\overset{3}{s}_2 \bar{\pi}_1^2(x)) = f_2 \bar{\pi}_2^3(\overset{3}{s}_i(x)). \end{aligned}$$

We order K_4 relative to K_3 in the same manner as we did K_3 relative to K_2, and make a similar definition of the local cross section

[3] For s in G and x in G/H, we define $sx = p(sp^{-1}(x))$ where p is the natural projection of G on G/H.

f_4. Continuing this process, we obtain the desired local cross sections, and hence, by Theorem 1, H has a local cross section.

G has an open set W which is a direct product, $W=ZA$, where $Z=\pi_1^{-1}(e)$ is the projective limit of the groups K_k, and A is a local Lie group. (See [**5**, pp. 214–215].) The set $U=ZV$, where V is the limit of V_k^*, is an open set contained in W (or may be so chosen). The set $R=ST$, where S is a cross section set of Z/Y, $Y=H\cap Z$, and T is the projective limit of $f_k(U_k^*)$, is a cross section set of U by our construction. If H is closed, then $B=A\cap H$ is a local Lie group such that $W\cap H=YB$. One may note from our construction that T is a cross section set for B in A. Hence, we have the following result.[4]

Corollary 1. *Let G be a group satisfying the hypothesis of Theorem 2. Let $W=ZA$, where Z is 0-dimensional, compact subgroup of G, A a local Lie group. If $H\cap W=YB$ is the corresponding decomposition of the open set $H\cap W$, then G/H is locally the direct product of Z/Y and A/B.*

Corollary 2. *Let G be a locally compact group which is separable, metric, and of finite dimension, and H a closed subgroup of G. Then G is a fibre bundle over G/H.*

Proof. See [**7**, p. 31].

References

1. C. Chevalley, *Theory of Lie groups*, Princeton University Press, 1946.

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3. A. M. Gleason, *Groups without small subgroups*, Ann. of Math. vol. 56 (1952) pp. 193–212.

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6. L. Pontrjagin, *Topological groups*, Princeton University Press, 1946.

7. N. Steenrod, *Theory of fibre bundles*, Princeton University Press, 1951.

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[4] One may observe that this decomposition enables us, using Theorem 1 and Proposition 1, p. 110 of [1], to construct a local cross section for H in a slightly different way. This fact and Corollary 1 were pointed out to the author by the referee.