ON THE REPRESENTATION OF AN INTEGER AS THE SUM OF THREE FOURTH POWERS

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1. **Introduction.** Let $r_{s,k}(N)$ denote the number of representations of a positive integer N as the sum of s kth powers of positive integers. S. Chowla and S. S. Pillai [2] proved that, for infinitely many N,

$$r_{k,k}(N) > c_1 \log \log N$$

where c_1 (as well as $c_2, c_3, \cdots$ in what follows) is a constant independent of N. P. Erdös [5] improved this and proved in an elementary way that, for infinitely many N,

$$r_{k,k}(N) > \exp\left(c_2 \frac{\log N}{\log \log N}\right).$$

However, for $s < k$ only particular cases have been studied. The only results I have come across are the following two due respectively to S. S. Pillai and S. Chowla (see Chowla [1, p. 122]). For an infinity of N, n,

$$r_{2,3}(N) > c_3 \log \log N$$

and

$$r_{3,4}(n) > c_4 \frac{\log n}{\log \log n}.$$

In this note (§3) I prove that, for infinitely many N,

$$r_{3,4}(N) > \exp\left(c_5 \frac{\log N}{\log \log N}\right).$$

The proof is elementary and entirely different from that of Chowla.

2. **Lemmas.** For the proof of our theorem we require some lemmas. In what follows, all the letters denote positive integers.

LEMMA 1. *Let $p_1, p_2, \cdots, p_r$ be the r consecutive primes >78 and belonging to the arithmetic progression $6n+1$. Then, corresponding to every p_i, there is a constant $k_i < 22464$ such that $n_i^2 = (p_i k_i)^2$ has a representation of the form $x_i^2 + 3y_i^2$ satisfying the conditions that $x_i > 7y_i > 0$ and p_i does not divide $x_i y_i$.*

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PROOF. It is well known (Euler [4]) that every prime of the form $6n+1$ is of the form x^2+3y^2. Let $p_i=\xi_{i1}^2+3\eta_{i1}^2$. The inequalities which follow are easily seen to be strict inequalities.

In case $\xi_{i1}>2\eta_{i1}$ we write $q_i=p_i$; while in case $\xi_{i1}<2\eta_{i1}$ we write $q_i=p_i(\lambda^2+3)$ where λ is the integer such that

$$\frac{\xi_{i1}}{\eta_{i1}} < \lambda < \frac{\xi_{i1}}{\eta_{i1}} + 1 < 3. \tag{2.1}$$

The number q_i defined in the second case has the representation

$$q_i = \xi_{i2}^2 + 3\eta_{i2}^2 \text{ (say)}$$

where

$$\xi_{i2} = \lambda\xi_{i1} + 3\eta_{i1} \quad \text{and} \quad \eta_{i2} = \lambda\eta_{i1} - \xi_{i1}. \tag{2.2}$$

From (2.1) and (2.2) we get

$$\xi_{i2} > 2\eta_{i2}. \tag{2.3}$$

We see from (2.3) that the representation

$$q_i^2 = \xi_{i3}^2 + 3\eta_{i3}^2, \qquad 0 < \xi_{i3} < \eta_{i3}, \tag{2.4}$$

is possible, q_i being defined by either $q_i=p_i$ or $q_i=p_i(\lambda^2+3)$. Hence

$$16q_i^2 = (6\eta_{i3} - 2\xi_{i3})^2 + 3(2\xi_{i3} + 2\eta_{i3})^2 = \xi_{i4}^2 + 3\eta_{i4}^2 \text{ (say)}. \tag{2.5}$$

From (2.4) and (2.5) we get

$$0 < \eta_{i4} < \xi_{i4} < 3\eta_{i4}.$$

Now

$$144(\xi_{i4}^2 + 3\eta_{i4}^2) = (6\xi_{i4} + 18\eta_{i4})^2 + 3(6\xi_{i4} - 6\eta_{i4})^2 = \xi_{i5}^2 + 3\eta_{i5}^2 \text{ (say)}.$$

We verify that

$$\xi_{i5} > 3\eta_{i5}.$$

We now consider two cases.

(i) $\xi_{i5}>7\eta_{i5}$. In this case we write $n_i=48q_i$.

(ii) $7\eta_{i5}>\xi_{i5}>3\eta_{i5}$. Now

$$2704(\xi_{i5}^2 + 3\eta_{i5}^2) = (46\xi_{i5} + 42\eta_{i5})^2 + 3(46\eta_{i5} - 14\xi_{i5})^2.$$

We can verify that

$$46\xi_{i5} + 42\eta_{i5} > 7\,|\,46\eta_{i5} - 14\xi_{i5}\,|.$$

In case (ii) we write $n_i=2496q_i$. The number n_i defined as above for the two cases is such that n_i^2 has a representation $x_i^2+3y_i^2$, $x_i>7y_i>0$. Using the fact that $p_i>78$ and does not divide $\xi_{i1}\eta_{i1}$ we can show that p_i does not divide x_iy_i.

LEMMA 2. *Let n_i, $i=1, 2, \cdots, r$, be as in Lemma 1. Then the number $n_{j_1}^2 n_{j_2}^2 \cdots n_{j_k}^2$, $0<j_1<j_2<\cdots<j_k\leqq r$, $1\leqq k<r$, has a representation of the form x^2+3y^2 with $x>7y>0$ and such that p_{j_s}, $s=1, 2, \cdots, k$, does not divide xy.*

The proof is by induction on k.

LEMMA 3. *Let $R_1(N)$ denote the number of representations of an integer N in the form x^2+3y^2, $x>7y>0$. Then, for infinitely many N,*

$$R_1(N) > \exp\left(c_6 \frac{\log N}{\log \log N}\right).$$

PROOF. Let

$$A = n_1 n_2 \cdots n_r,$$
$$B = n_{j_1} n_{j_2} \cdots n_{j_k}, \qquad 0 < j_1 < j_2 < \cdots < j_k \leqq r,\ 1 \leqq k < r,$$
$$A = BC \quad \text{and} \quad N = A^2,$$

where the n's are as in Lemma 1.

By Lemma 2, B has a representation x^2+3y^2, $x>7y>0$, such that p_{j_s}, $s=1, 2, \cdots k$, does not divide xy. This gives a representation $(Cx)^2+3(Cy)^2$ for N satisfying the conditions of Lemma 3. No two B's can give rise to the same representation. For, if B and B' give the same representation we shall have an equality of the type $Cx=C'x'$. If $C\neq C'$ there is a prime p which divides (say) C but not C'. So p divides x'. Also p divides B' since it does not divide C'. This is a contradiction since the representation of B' which we consider is such that if p divides B' then p does not divide x'.

Hence

$$R_1(N) \geqq \sum_{B} 1 = 2^r - 2. \tag{2.6}$$

Now, by the prime number theorem for arithmetic progressions, $p_k<c_7 k \log k$ and

$$\log N = 2(\log n_1 + \cdots + \log n_r) < c_8 r \log r,$$

so that

$$r > c_9 \frac{\log N}{\log \log N}. \tag{2.7}$$

From (2.6) and (2.7) the conclusion of the lemma follows.

LEMMA 4. *Let* $R_2(N)$ *denote the number of representations of an integer* N *in the form* $\xi^2-\xi\eta+\eta^2$, $\xi>4\eta>0$. *Then*

$$R_2(N) > \exp\left(c_6 \frac{\log N}{\log \log N}\right)$$

for infinitely many N.

PROOF. Writing $\xi=x+y$ and $\eta=2y$ and taking N as in Lemma 3, we get

$$R_2(N) = R_1(N) > \exp\left(c_6 \frac{\log N}{\log \log N}\right).$$

3. **Theorem.** *For infinitely many* M,

$$r_{3,4}(M) > \exp\left(c_5 \frac{\log M}{\log \log M}\right).$$

PROOF. Consider the identity [3]

$$(k^2-2k)^4+(2k-1)^4+(k^2-1)^4 = 2(k^2-k+1)^4.$$

Writing $k=\xi/\eta$ and multiplying by η^8 this becomes

$$(3.1) \quad (\xi^2-2\xi\eta)^4+(2\xi\eta-\eta^2)^4+(\xi^2-\eta^2) = 2(\xi^2-\xi\eta+\eta^2)^4.$$

If N is as in Lemmas 3 and 4, there are $R_2(N)$ pairs of values of ξ and η, $\xi>4\eta>0$, for which the right-hand side of (3.1) can be fixed equal to $M=2N^4$. Thus we get $R_2(N)$ representations of N in the the form $x_1^4+x_2^4+x_3^4$.

Now if $\xi>4\eta$ the greatest and the smallest of the expressions on the left-hand side of (3.1) are respectively $\xi^2-\eta^2$ and $2\xi\eta-\eta^2$. If ξ_1, η_1 and ξ_2, η_2 give the same representation, then

$$\xi_1^2-\xi_1\eta_1+\eta_1^2 = \xi_2^2-\xi_2\eta_2+\eta_2^2,$$

$$\xi_1^2-\eta_1^2 = \xi_2^2-\eta_2^2,$$

and

$$2\xi_1\eta_1-\eta_1^2 = 2\xi_2\eta_2-\eta_2^2.$$

These relations are easily seen to be inconsistent unless $\xi_1=\xi_2$ and $\eta_1=\eta_2$. Hence all the representations given by different pairs of values of ξ and η are distinct.

It follows that

$$\begin{aligned} r_{3,4}(M) &\geqq R_2(N) \\ &> \exp\left(c_6 \frac{\log N}{\log \log N}\right) \\ &> \exp\left(c_5 \frac{\log M}{\log \log M}\right). \end{aligned}$$

References

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