

EXTREME POINTS OF VECTOR FUNCTIONS

SAMUEL KARLIN

Several previous investigations have appeared in the literature which discuss the nature of the set T in n -dimensional space spanned by the vectors $(\int_S d\mu_1, \dots, \int_S d\mu_n)$ where S ranges over all measurable sets. It was first shown by Liapounov that if μ_1, \dots, μ_n are atomless measures, then the set T is convex and closed. Extensions of this result were achieved by D. Blackwell [1] and Dvoretzky, Wald, and Wolfowitz [2]. A study of the extreme points in certain function spaces is made from which the theorems concerning the range of finite vector measures are deduced as special cases. However, the extreme point theorems developed here have independent interest. In addition, some results dealing with infinite direct products of function spaces are presented. Many of these ideas have statistical interpretation and applications in terms of replacing randomized tests by nonrandomized tests.

1. Preliminaries. Let μ denote a finite measure defined on a Borel field of sets \mathfrak{F} given on an abstract set X . Let $L(\mu, \mathfrak{F})$ denote the space of all integrable functions with respect to μ . Finally, let $M(\mu, \mathfrak{F})$ be the space of all essentially bounded measurable functions defined on X . It is well known that $M(\mu, \mathfrak{F})$ constitutes the conjugate space of $L(\mu, \mathfrak{F})$ and consequently the unit sphere of $M(\mu, \mathfrak{F})$ is bi-compact in the weak * topology [3]. Let $(\otimes L^n)$ denote the direct product of L taken with itself n times and take $(\otimes M^n)$ as the direct product of M n times. With appropriate choice of norm, $(\otimes M^n)$ becomes the conjugate space to $(\otimes L^n)$. Another description of $(\otimes M^n)$ is that it consists of all n -vectors, each component of which is an essentially bounded measurable function. In notation, let $x \in (\otimes M^n)$ be denoted by $\bar{x} = (x_i(t))$, $i = 1, \dots, n$, with x_i in M .

2. Extreme points in M_A . Let A be a bounded closed convex set in Euclidean n -space. Let M_A consist of all x in $(\otimes M^n)$ whose range of values lie almost everywhere in A . I.e., for almost every t , $\bar{x}(t) = (x_i(t))$ is in A . Let B consist of the extreme points of A , and let \bar{B} = closure of B . For $n \geq 3$, it is not necessarily true that $\bar{B} = B$.

In view of the convexity of A , the set M_A is easily verified to be convex, bounded, and weak * closed. We indicate the proof of this last fact. Let \bar{x}_0 represent a weak * limit point of M_A . Suppose $\bar{x}_0 \notin M_A$. This implies the existence of a set of positive finite measure

Received by the editors October 11, 1952 and, in revised form, October 30, 1952.

E with $\bar{x}_0(E) \notin A$. Since a denumerable number of hyperplanes determine A , it is easy to construct a plane $\{\xi_i\}$ so that $\sum_{i=1}^n \xi_i \eta_i > c$ for $\{\eta_i\} \in A$ while $\sum_{i=1}^n \xi_i x_i^0(E_1) < c - \epsilon$ for a set of positive measure $E_1 \subset E$. Let $\bar{w}(t)$ denote the function in $\otimes L^n(\mu, \mathfrak{F})$ defined as follows:

$$\bar{w}(t) = \left(w_i(t) = \begin{cases} \xi_i/\mu(E_1) & \text{for } t \text{ in } E_1 \\ 0 & \text{for } t \notin E_1 \end{cases} \right).$$

We note that $(\bar{w} \cdot \bar{x}) = \sum_{i=1}^n \int w_i x_i d\mu$ represents the inner product and is $\geq c$ for \bar{x} in M_A , while $(\bar{w} \cdot \bar{x}_0) < c - \epsilon$. This shows that \bar{x}_0 cannot be a weak * limit point of M_A , a contradiction from which we conclude that $\bar{x}_0 \in M_A$.

Since the unit sphere $(\otimes M^m)$ is bicomact in the weak * topology, we deduce that M_A is bicomact. On account of the bicomactness and convexity, the Krein-Milman theorem guarantees the existence of extreme points in M_A . The first theorem characterizes such extreme points. We define $M_{\bar{B}}$ in a similar manner to M_A .

THEOREM 1. *The extreme points of M_A are contained in $M_{\bar{B}}$.*

PROOF. Let $\bar{x}_0 = (x_i^0)$ be an extreme point of M_A and let us suppose the contrary that \bar{x}_0 is not in $M_{\bar{B}}$. There exists an ϵ_0 such that $\bar{x}_0 \notin M_{\bar{B}}(\epsilon_0)$ where $\bar{B}(\epsilon_0)$ consists of the closure of the set of points of A obtained from \bar{B} by describing an ϵ_0 sphere about every point in B . In fact, otherwise, let $\epsilon_n \rightarrow 0$ and suppose $\bar{x}_0 \in M_{\bar{B}}(\epsilon_n)$ for every n . That is, except for a set E_n of μ measure zero the range of values of \bar{x}_0 is in $M_{\bar{B}}(\epsilon_n)$. Since $\mu(\sum E_n) = 0$, we deduce that \bar{x}_0 is in $M_{\bar{B}} = M_{\bar{B}}(\epsilon_n)$, which serves as a contradiction. Thus there exists a set E of positive measure such that $\bar{x}_0(t)$, for t in E , is not in $\bar{B}(\epsilon_0)$. It follows easily that a constant vector \bar{a} exists of small magnitude such that $\bar{x}_0 \pm \bar{a}$ is in A when t is in $E_0 \subset E$. Put

$$\bar{\phi}(t) = \begin{cases} \bar{a}, & t \in E_0, \\ 0, & t \notin E_0. \end{cases}$$

Consequently, $\bar{x}_0 = (\bar{x}_0 + \bar{\phi})/2 + (\bar{x}_0 - \bar{\phi})/2$ with $\bar{x}_0 + \bar{\phi}$, $\bar{x}_0 - \bar{\phi}$ in M_A . This impossibility establishes the result, Q.E.D.

REMARK. The set $M_{\bar{B}}$ need not be contained in $(M_A)^{(e)}$ (extreme points) but M_B is.

3. Extreme points of M_A with side conditions. A finite measure μ is said to be atomless if for any measurable set E of positive measure μ there exists a measurable subset E_0 such that $0 < \mu(E_0) < \mu(E)$. The theorem of Liapounov [4] states that if μ_1, \dots, μ_n are atom-

less finite measures defined on the same Borel field of sets, then the span in Euclidean n space of the n tuples $(\int x(t)d\mu_j(t))$ obtained by allowing $x(t)$ to range over all characteristic functions is convex and closed. Therefore, it follows that the set of all n tuples $(\int x(t)d\mu_j(t))$ with $0 \leq x(t) \leq 1$ spans the same set. An apparent weaker formulation of the above spanning result which is, however, equivalent is that for any given measurable set S there exists a set $E \subset S$ such that simultaneously $\mu_i(E) = \mu_i(S)/2$ for $i=1, \dots, n$.

Let $M[A, \mu_j] = [\xi_i, i=1, \dots, n]$ with $\xi_i = \int \bar{x}(t)d\mu_i(t)$ and \bar{x} in M_A . It is of course understood that x is measurable with respect to all μ_j . This set can also be viewed as points in Euclidean nm space.

THEOREM 2. *If μ_i ($i=1, \dots, n$) consist of atomless finite measures, then the extreme points of the set T of all $\bar{x}(t)$ measurable μ_i ($i=1, \dots, n$) satisfying $\bar{x} \in M_A$ and $\bar{b}_i \leq \int \bar{x}d\mu_i \leq \bar{a}_i$ are contained in $M_{\bar{B}}$.*

REMARK 1. The vector inequality $\int \bar{x}d\mu_i \leq \bar{a}_i$ means that the inequality holds for each component of the vectors.

REMARK 2. The interest of Theorem 2 is that if we impose linear side conditions on the set M_A of the form $\bar{b}_i \leq \int \bar{x}d\mu_i \leq \bar{a}_i$, then no new extreme points are added to this convex subset of M_A when $B = \bar{B}$. Only the set of extreme points of M_A may be diminished. This result is in sharp contrast to the situation in the case of atomic measures. We leave it to the reader to construct examples involving atomic measures which violate the conclusions of the theorem.

REMARK 3. If $\mu = \mu_1 + \dots + \mu_n$, then by virtue of the Radon-Nikodým theorem $\mu_i(E) = \int_E f_i(t)d\mu(t)$. The space $M(\mu, \mathfrak{F})$ considered here is the set of all bounded measurable functions $x(t)$, which is the conjugate space to $L(\mu, \mathfrak{F})$ where \mathfrak{F} is the common Borel field of sets over which all the μ_i are defined. Therefore, bounded weak* closed sets in $M(\mu, \mathfrak{F})$ are bicomact and the same holds for $(\otimes M(\mu, \mathfrak{F}))^n$. The statement of the theorem in terms of the f_i is that the set Γ of extreme points of all $\bar{x}(t)$ in M_A which satisfy $\bar{b}_i \leq \int \bar{x}f_i(t)d\mu(t) \leq \bar{a}_i$ is contained in $M_{\bar{B}}$ when μ is atomless. Since A is convex and closed and the linear restrictions are generated by elements of $L(\mu, \mathfrak{F})$, we deduce that T is bicomact and convex and hence possesses extreme points. The theorem thus determines the form of such extreme points.

PROOF OF THEOREM 2. Let \bar{x}_0 be an extreme point of T and suppose \bar{x}_0 is not in $M_{\bar{B}}$. Then an argument as in Theorem 1 shows that for some positive ϵ_0 the point \bar{x}_0 is not in $M_{\bar{B}(\epsilon_0)}$. Consequently there exists a set S of positive measure μ for which $x_0(t) \notin \bar{B}(\epsilon_0)$ for t in S . As in Theorem 1, one can find a constant vector \bar{c} and a measurable set $E \subset S$ of positive μ measure such that $\bar{x}_0 \pm \bar{c}$, for t in E , is contained

in A . An application of the theorem of Liapounov yields a measurable subset E_0 of E with

$$\mu_i(E_0) = \int_{E_0} f_i d\mu = \frac{1}{2} \int_E f_i d\mu = \frac{1}{2} \mu_i(E) \quad \text{for } i = 1, \dots, n.$$

Put

$$\phi(t) = \begin{cases} \bar{c}, & t \text{ in } E_0, \\ -\bar{c}, & t \text{ in } E - E_0, \\ 0 & \text{elsewhere,} \end{cases} \quad \text{and} \quad \psi(t) = \begin{cases} -\bar{c}, & t \text{ in } E_0, \\ \bar{c}, & t \text{ in } E - E_0, \\ 0 & \text{elsewhere.} \end{cases}$$

It is easy to verify that $\bar{x}_0 = (\bar{x}_0 + \phi)/2 + (\bar{x}_0 + \psi)/2$ where $\bar{x}_0 + \phi$ and $\bar{x}_0 + \psi$ are in M_A . A simple computation gives that

$$\begin{aligned} \int (\bar{x}_0 + \phi) f_i d\mu &= \int \bar{x}_0 f_i d\mu + \bar{c} \left[\int_{E_0} f_i d\mu - \int_{E-E_0} f_i d\mu \right] \\ &= \int \bar{x}_0 f_i d\mu + \bar{c} \left[\frac{1}{2} \int_E f_i d\mu - \frac{1}{2} \int_E f_i d\mu \right] \\ &= \int \bar{x}_0 f_i d\mu. \end{aligned}$$

Similarly, $\int (\bar{x}_0 + \psi) f_i d\mu = \int \bar{x}_0 f_i d\mu$. Therefore $\bar{x}_0 + \psi$ and $\bar{x}_0 + \phi$ satisfy the linear inequalities and hence lie in T . We have exhibited a contradiction of the extreme point nature of \bar{x}_0 and the proof of the theorem is hereby complete.

Although we employed the convexity part of the theorem of Liapounov, we can now use Theorem 2 to obtain an extension of the theorem of Liapounov. It is to be remarked, however, that the proof of convexity in the Theorem of Liapounov is the simpler result to obtain and is also used in [1] and [2].

THEOREM 3. *Let μ_j ($j = 1, \dots, n$) be atomless, finite measures. Then $M[A, \mu_j] = M[\bar{B}, \mu_j]$.*

REMARK 4. An immediate conclusion of this theorem is that the range in E^{nm} of $\int \bar{x}(t) d\mu_j(t)$ where $\bar{x}(t)$ ranges over $M_{\bar{B}}$ is convex and closed. This follows from the evident convexity and closedness of $M[A, \mu_j]$ (see Remark 3).

PROOF. Let \bar{x}_0 be any point in M_A . Put $\bar{\xi}_i = \int \bar{x}_0 d\mu_i$. Let Γ be the set of all \bar{x} in M_A for which $\bar{\xi}_i = \int \bar{x} d\mu_i$. The set is weak * closed and convex and hence bicomact (see Remark 3). There exists consequently an extreme point \bar{x}_1 of Γ . By virtue of Theorem 2, \bar{x}_1 is in

M_B and $\int \bar{x}_1 d\mu_i = \xi_i = \int \bar{x}_0 d\mu_i$. Thus, we have shown that any point in $M[A, \mu_j]$ is in $M[\bar{B}, \mu_j]$. On the other hand since $\bar{B} \subset A$, evidently we get $M[\bar{B}, \mu_j] \subset M[A, \mu_j]$. Combining we have established our result.

COROLLARY 1 [1]. *Let C represent any closed bounded set in E^m , then $M[C, \mu_i]$ is convex and closed if the μ_j are atomless.*

PROOF. Let A be the convex span of C . Clearly A is closed and $\bar{B} \subset C$. Evidently, $M[\bar{B}, \mu_i] \subset M[C, \mu_i] \subset M[A, \mu_i]$. However, Theorem 3 implies the equality of the outside two sets which thus gives the result (see Remark 4).

As was pointed out in [1] and [2] the closure of $M[C, \mu_i]$ can be established under any circumstances regardless of the nature of the measures μ_i . In fact, any measure μ can be expressed as a sum of an atomless measure μ^* and a countable union of pure atomic measures which we designate by $\bar{\mu}$. In notation, $\mu = \mu^* + \bar{\mu}$. In the case of $\bar{\mu}$ one can show directly the closure [1; 2]. The case of μ^* was handled in Theorem 3. Thus, to obtain the closure property for the general case one needs only to apply this decomposition result to the measure $\mu = \mu_1 + \mu_2 + \dots + \mu_n$, and invoke the above remarks to the parts of μ . Precisely: $M[C, \mu^* + \bar{\mu}] = \text{convex span}(M[C, \mu^*], M[C, \bar{\mu}])$.

Closure properties of extreme points. This section investigates whether the set of extreme points of M_A is weak * closed or weakly closed. The measure μ is taken to be atomless and A is assumed to contain more than one point.

THEOREM 4. *Let $B = \bar{B}$, then the set of extreme points of M_A is not weak * closed and the extreme points M_B and M_A are sequentially weakly closed.*

PROOF. We suppose that the set of extreme points of M_A is weak * closed. Let \bar{x} denote any element of M_A and consider any integrable $f_1(t), \dots, f_n(t)$. Theorem 2 provides an extreme point \bar{x}_0 such that $\int f_i \bar{x} d\mu = \int f_i \bar{x}_0 d\mu$ for $i = 1, \dots, n$. Of course, \bar{x}_0 depends on the choice of f_1, \dots, f_n . Put $G(f_\alpha) = [\bar{x}_0 | \bar{x}_0 \text{ an extreme point of } M_A \text{ and } \int \bar{x}_0 f_\alpha d\mu = \int \bar{x} f_\alpha d\mu]$. The assumption implies that $G(f_\alpha)$ is weak * closed and nonempty. The above argument shows that every finite intersection of $G(f_{\alpha_i})$ is weak * closed and nonempty. Since the extreme points of M_A are weak * closed and hence bicomact, we have $L = \bigcap G(f_\alpha) \neq \emptyset$. Let \bar{x}_0 be in L , then $\int \bar{x} f d\mu = \int \bar{x}_0 f d\mu$ for every integrable f and hence $\bar{x} = \bar{x}_0$ almost everywhere. Hence every \bar{x} in M_A is a point in $M_{\bar{B}}$, which is clearly impossible since A consists of more than one point. We now verify the second assertion of the theorem. If \bar{x}_n con-

verges weakly to \bar{x} , then at least \bar{x}_n converges almost everywhere to \bar{x} and $\|\bar{x}_n\| < C$. Thus if \bar{x}_n are in M_B , then clearly \bar{x} is in M_B as B is closed. This completes the proof of the theorem.

REMARK 5. The same result concerning the extreme points can be carried over to sets studied in Theorem 2. Furthermore, in the case that μ is purely atomic and $B = \bar{B}$, it is easily seen that the set of extreme points M_B of M_A is weak * closed. We omit the details. This is in contrast with Theorem 4.

A further example more clearly illustrating the conclusion of Theorem 4 is now given explicitly. On the basis of Theorem 1, it follows that the extreme points of the set of all Lebesgue measurable positive functions bounded by one consist of all characteristic functions. In particular, $s_n(t) = 1/2 + r_n(t)/2$, where $r_n(t) = \text{sign} \sin 2^{n+1}\pi t$, is an extreme point for each n , where $r_n(t)$ are the classical Rademacher functions. The orthogonality and boundedness of $r_n(t)$ imply for any integrable function $f(t)$ that $\lim_{n \rightarrow \infty} \int_0^1 f(t)r_n(t)dt = 0$. Consequently $s_n(t)$ converges weak * to the function identically equal to $1/2$ which is clearly not an extreme point.

Infinite vector functions. The space $M^\infty = (\otimes M(\mu, \mathfrak{F}))^\infty$ is defined as the countable infinite direct product of $M(\mu, \mathfrak{F})$. An element of M^∞ has the form $\bar{x} = (x_1(t), x_2(t), \dots)$ where each $x_i(t)$ is in $M(\mu, \mathfrak{F})$. Let (m) denote the Banach space consisting of all bounded sequences. It is well known that (m) is the conjugate space of (l) (sequences which converge absolutely). Therefore bounded weak * closed sets in (m) are bicomact and convex bounded weak * closed sets are spanned in the weak * topology by the extreme points. This is a restatement of the Krein-Milman Theorem.

We discuss an example to indicate the extensions of the preceding theory to the infinite case.

A. Let $\bar{x} = (x_i(t))$, $i = 1, 2, \dots$, with $0 \leq x_i(t) \leq 1$. Let M_A denote the set of all such points in M^∞ . It can be shown in a manner similar to the proof of Theorem 1 that the extreme points of M_A consist of those elements \bar{x} whose value for almost every t lie in the extreme points of the set B in (m) which consists of sequences (m_i) with $m_i = 0$ or 1 . Also M_A is weak * closed. Furthermore, it follows that if additional linear conditions $\int \bar{x} d\mu_j = \bar{\alpha}_j$, $j = 1, \dots, m$, are imposed with μ_j atomless, then no new extreme points are added. This yields as in Theorem 3 the convexity and weak * closure of the span in (m) of $(\int \bar{x} d\mu_j)$.

More detailed results on the infinite vector functions will appear in a later publication.

Extreme points in measure spaces. It is of interest to compare the type of results obtained above with the description of the set of extreme points in other Banach spaces where constraints are present. Specifically, we study the extreme points of the set of all positive regular measures of total variation one defined on a bi-compact Hausdorff space X subject to linear constraints.

Let $h_1(x), \dots, h_n(x)$ be continuous functions defined on X . We look to characterize the set of extreme points of the set L of all positive measures = distributions μ with $\int d\mu = 1$, $\int h_i(x)d\mu = a_i$. If we consider the image M in Euclidean n -space of $\mu \rightarrow \{\int h_i(x)d\mu\}$, then it follows easily that M consists of the set of all points, in the convex span of the bi-compact image Γ of X given in parametric form by $t_i = h_i(x)$, $i = 1, \dots, n$, with x in X . Indeed, let C represent the convex span of Γ . As the convex span of the compact set Γ , C is closed. If \bar{i} denotes a point in M but outside C , then there exists a hyperplane which separates C and \bar{i} . Therefore constants b_i exist so that $\sum b_i \bar{t}_i < -\delta < 0$ and $\sum b_i h_i(x) \geq 0$ for each x in X . Since $\bar{t}_i = \int h_i(x)d\mu(x)$ for some μ , then $0 > \sum b_i \bar{t}_i = \int \sum b_i h_i(x)d\mu(x) \geq 0$, which is impossible. This shows that $C = M$.

We now specialize the functions $h_i(x)$ so that for any choice of n points x_1, x_2, \dots, x_n the rank of the matrix $h_i(x_j)$ is n . Under this condition we now show that points on the boundary of M come from unique measures possessing only a finite number of points of increase. To this end, let \bar{i} be a point on the boundary of M whose components are given by $t_i = \int h_i(x)d\mu_0(x)$. There exists a supporting plane to M at \bar{i} . Hence, $\int \sum b_i h_i(x)d\mu_0(x) = 0$ for appropriate constants and, since $\sum b_i h_i(x) \geq 0$, we get for every point x of increase of μ_0 that $\sum b_i h_i(x) = 0$. This implies in view of the property of $h_i(x)$ that at most n points of increase are possessed by μ_0 . Since $\sum_{j=1}^n \lambda_j h_j(x_i) = t_i$ where $h_i(x_j)$ has rank n we obtain that the λ_j are unique. This establishes the assertion that boundary points of M correspond to unique measures.

Finally, we investigate the form of the set of extreme points of L . It is clear that the linear constraints $\int h_i(x)d\mu(x) = a_i$ define geometrically a section K of the convex set M and that extreme points of K must be boundary points of M . Thus any extreme point of K is built up of a convex finite sum of pure atomic measures. Without loss of generality, we can therefore restrict our considerations only to measures μ which are finite convex combinations of pure atomic measures. Let $h_0(x) \equiv 1$. We now suppose, in addition to the conditions stated before, that the rank of the determinant $h_i(x_j)$ for $n+1$ distinct points x_j is $n+1$, $i=0, 1, \dots, n$. Suppose now that μ

$= \sum \alpha_i \mu_i$ with μ_i in L ($\alpha_i \geq 0$). Clearly any point of increase for a single μ_i is shared by μ . Let x_1, \dots, x_m denote the totality of points of increase of μ . Since $\int h_i(x) d\mu(x) = a_i$ for $i = 1, \dots, n$ and $\int h_0(x) d\mu(x) = 1$, we obtain that

$$(*) \quad \sum_{i=1}^m \lambda_i h_j(x_i) = a_j, \quad j = 0, 1, \dots, n,$$

where $a_0 = 1$. Since the rank of $[h_j(x_i)]$ is $\min(n+1, m)$ it follows that the dimensionality of solutions in λ of the linear equations (*) will be $m - \min(n+1, m) = \max(m - (n+1), 0)$. Thus if $m \leq n+1$, then the solution is unique and μ is an extreme point. If $m > n+1$, then since a solution with $\lambda_i > 0$ exists, it follows if $\sum_{i=1}^m z_i h_j(x_i) = 0$ that $\lambda_i \pm \epsilon z_i \geq 0$ and $\sum (\lambda_i \pm \epsilon z_i) h_j(x_i) = a_j$ for ϵ sufficiently small and hence $\lambda_i = (\lambda_i + \epsilon z_i)/2 + (\lambda_i - \epsilon z_i)/2$. This shows that the extreme points are given by precisely those measures of L which are composed of not more than $n+1$ pure atomic measures. We have thus established:

THEOREM 5. *The extreme points of the set L of all positive regular measures over a topological space X satisfying $\int d\mu(x) = 1$, $\int h_i(x) d\mu = a_i$, $i = 1, \dots, n$, with $h_i(x)$ continuous, consist of those measures in L with not more than $n+1$ points of increase provided that the rank of $\{h_i(x_j)\}$ is $n+1$ where $i, j = 0, 1, \dots, n$, $h_0(x) \equiv 1$, and x_j are distinct points of X .*

REFERENCES

1. D. Blackwell, *The range of a vector measure*, Proceedings of the American Mathematical Society vol. 2 (1951).
2. A. Dvoretzky, A. Wald, and J. Wolfowitz, *Relations among certain ranges of vector measures*, Pacific Journal of Mathematics (1951).
3. L. Alaoglu, *Weak topologies of normed linear spaces*, Ann. of Math. (1940).
4. A. Liapounov, *Sur les fonctions-vecteurs complement additives*, Bull. Acad. Sci. USSR. vol. 4 (1940).

CALIFORNIA INSTITUTE OF TECHNOLOGY