VARIOUS REMARKS ON UNIVALENT FUNCTIONS

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1. Let S denote the class of functions $f(z)$ regular and univalent for $|z|<1$

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots + a_nz^n + \cdots .$$

Let V_n denote the nth coefficient region for this class of functions [5, §1.2]. Let $F=F(a_2, \bar{a}_2, \cdots, a_n, \bar{a}_n)$ be a real-valued function. Write $x_\nu=(1/2)(a_\nu+\bar{a}_\nu)$, $y_\nu=(1/2i)(a_\nu-\bar{a}_\nu)$, $F_\nu=(1/2)(\partial F/\partial x_\nu - i\partial F/\partial y_\nu)$. Let F satisfy the further conditions

(a) F is defined in an open set O containing V_n,

(b) F and its derivatives F_ν are continuous in O,

(c) $|\text{grad } F| =(\sum_{\nu=2}^{n} |F_\nu|^2)^{1/2}>0$ in O.

Let $f(z)^k= \sum_{\nu=k}^{\infty} a_\nu^{(k)}z^\nu$. Then the following is one of the basic results of Schaeffer and Spencer [5, Lemma VII].

I. Every function $f(z)$ of class S belonging to a point $(a_2, a_3, \cdots, a_n)$ where F attains its maximum in V_n must satisfy the differential equation

$$\left(z\frac{f'(z)}{f(z)}\right)^2 \sum_{\nu=1}^{n-1} \frac{A_\nu}{f(z)^\nu} = \sum_{\nu=-(n-1)}^{n-1} \frac{B_\nu}{z^\nu} \tag{1}$$

where

$$A_\nu = \sum_{k=\nu+1}^{n} a_k^{(\nu+1)}F_k,\quad B_\nu = \sum_{k=1}^{n-\nu} ka_k F_{k+\nu}, \qquad \nu = 1, 2, \cdots, n-1,$$

$$B_0 = \sum_{k=1}^{n} (k-1)a_k F_k,\quad B_{-\nu} = \overline{B}_\nu,$$

the derivatives being taken at the point $(a_2, \cdots, a_n)$. Also $B_0>0$ and the right-hand side of (1) is non-negative on $|z|=1$ with at least one zero there.

While consideration of the above class of functions F proved very advantageous in the discussion of qualitative properties of V_n given by the above mentioned authors, as far as explicit bounds are concerned a second class of functions has in general attracted more attention. These are what we shall refer to as w-homogeneous polynomials in $a_2, \cdots, a_n$.

If $f(z)$ belongs to S, so does $e^{-i\theta}f(e^{i\theta}z)$ for all real θ. In the Taylor

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expansion of this function about $z=0$ the coefficient of z^n is $a_n e^{i(n-1)\theta}$. Let $P(a_2, \cdots, a_n)$ be a polynomial in $a_2, \cdots, a_n$. If

$$P(a_2 e^{i\theta}, \cdots, a_n e^{i(n-1)\theta}) = e^{ik\theta} P(a_2, \cdots, a_n)$$

where $k \neq 0$ we call P a w-homogeneous polynomial.

If we desire to find the maximum of the modulus of such a polynomial in V_n we may clearly assume that it is attained at a point where P is real and positive. Consideration of the proof of I then shows that the maximizing function $f(z)$ must satisfy the differential equation (1) where P_ν replacing F_ν is now actually the partial derivative of P with respect to a_ν. Otherwise all statements in I hold word for word.

For an explicitly given w-homogeneous polynomial P starting with the result I there are two principal methods of obtaining the desired bound. The first uses the existence of a zero on $|z| = 1$ of the right-hand side of (1) [**3**, p. 620; **4**, pp. 120, 122]. The second undertakes to integrate (1) directly and derive information from this [**3**, pp. 620–623; **5**, chap. XIV]. This second method is extremely complicated in all but the simplest cases.

The principal reason for this complication is that integration leads to equality between two rather involved functions, one depending on w the other on z. It is natural to inquire as to conditions under which this process can be applied with relative simplicity. The most obvious such situation is that where only a single term occurs on the left-hand side of (1), i.e., $A_\nu = 0$ for $\nu = 1, \cdots, n-2$. We shall now show that the only w-homogeneous polynomials for which this occurs are a_2, $a_2^2 - a_3$, and their powers (possibly multiplied by a coefficient, this qualification will always be understood in future), i.e. quantities for which bounds are obtained by elementary methods.

For $n=2$ it is evident that every w-homogeneous polynomial is a power of a_2 and for these only a single term occurs on the left-hand side of (1).

For $n=3$ the condition $A_1 = 0$ is seen at once to give $P_2 + 2a_2 P_3 = 0$. Elementary calculation shows that P is necessarily a function of $a_2^2 - a_3$. In order to be a w-homogeneous polynomial it must then be a power of this quantity.

Now suppose that P is a w-homogeneous polynomial involving a_n $(n > 3)$ but no higher coefficient. We shall show that the condition that only a single term occurs on the left-hand side of (1) leads to a contradiction. Indeed the conditions $A_{n-2} = 0$, $A_1 = 0$ written explicitly give

$$P_{n-1} + (n-1) a_2 P_n = 0,$$

$$P_2 + 2a_2P_3 + \cdots + (2a_{n-1} + \cdots)P_n = 0.$$

In the second equation a_{n-1} occurs only in the coefficient of P_n and there as $2a_{n-1}$, the remaining terms in the sum involving lower coefficients. Forming the Poisson bracket of these differential equations for P we obtain $(n-3)\, P_n = 0$. This implies that P does not involve a_n, contrary to assumption.

It should be observed that, the above result notwithstanding, there are w-homogeneous polynomials involving the higher coefficients for which precise bounds are known. For example, the coefficients of a function inverse to a function $f \in S$ when expressed in terms of the coefficients of f provide such polynomials.

2. Let Σ denote the class of functions $f(z)$ meromorphic and univalent for $|z| > 1$

$$f(z) = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$$

and let Σ^{-1} denote the class of functions $\phi(w)$ inverse to the preceding

$$\phi(w) = w + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots .$$

Springer [6] has derived for the coefficients a_i, b_i variational formulae analogous to those given by Schaeffer and Spencer. Let us consider first the class Σ^{-1}. If $\phi(w)$ belongs to this class, so does $e^{i\theta}\phi(e^{-i\theta}w)$ for real θ. The latter function has in its Laurent expansion about $w = \infty$ the coefficient $e^{i(n+1)\theta}b_n$ for z^{-n}. A polynomial $P(b_1, b_2, \cdots, b_n)$ will now be called w-homogeneous if

$$P(b_1e^{i2\theta}, \cdots, b_ne^{i(n+1)\theta}) = e^{ik\theta}P(b_1, \cdots, b_n)$$

where $k \neq 0$. Let $\phi(w)$ be a function for which $|P(b_1, \cdots, b_n)|$ attains its maximum. Then using the variational formula [6, p. 430] and reasoning as before (with the admissible assumption that the extremal function has $P(b_1, \cdots, b_n)$ real and positive) we find that the inverse function $f(z)$ must satisfy the differential equation

$$\left(z \frac{f'(z)}{f(z)}\right)^2 \sum_{\mu=2}^{n+1} C_\mu f(z)^\mu = Q_n(z) \tag{2}$$

where C_μ are certain expressions in $b_1, \cdots, b_n$ which can be read off from the variational formula and $Q_n(z)$ is a rational function of z. The latter has properties analogous to those of the function appearing on the right-hand side of (1). We shall again inquire for w-homogeneous polynomials such that the left-hand side of (2) involves just one term, i.e., $C_\mu=0$, $\mu=2, \cdots, n$.

For $n=1$ the situation is trivial: all w-homogeneous polynomials are powers of a_1.

For $n=2$ the condition $C_2=0$ reduces to $P_1=0$. Thus all polynomials in question are powers of a_2.

For $n=3$ the conditions $C_2=0$, $C_3=0$ are respectively $P_1-b_1P_3=0$, $P_2=0$. Elementary calculation shows that P is necessarily a power of $b_1^2+2b_3$. This last quantity is just the one considered by Springer in obtaining a precise bound for b_3. It was precisely the above conditions (together with certain properties of $Q_n(z)$) which enabled him to carry out the estimation.

For $n>3$ the conditions $C_2=0$, $C_{n-1}=0$ written explicitly give

$$- P_1 + b_1P_3 + 2b_2P_4 + \cdots + (n-2)b_{n-2}P_n = 0,$$
$$- P_{n-2} + b_1P_n = 0.$$

Forming the Poisson bracket of these differential equations for P we obtain $(n-3)P_n=0$. This shows that there is no w-homogeneous polynomial involving coefficients higher than b_3 and having the above property.

Now let us consider the class Σ. W-homogeneous polynomials in the a's are defined in the same manner as those in the b's. Let $f(z)$ be a function for which $|P(a_1, \cdots, a_n)|$ attains its maximum. Assuming, as we may, that the corresponding value of $P(a_1, \cdots, a_n)$ is real and positive and using the variational formula [6, p. 427] we find that the function $f(z)$ must satisfy a differential equation

$$\left(z\frac{f'(z)}{f(z)}\right)^2 \sum_{2}^{n+1} E_\mu f(z)^\mu = S_n(z) \tag{3}$$

where E_μ are certain expressions in $a_1, \cdots, a_n$ which can be read off from the variational formula and $S_n(z)$ is a rational function of z. We inquire for w-homogeneous polynomials such that the left-hand side of (3) involves just one term, i.e. $E_\mu=0$, $\mu=2, \cdots, n$.

An argument almost identical with the preceding shows that for $n=1, 2, 3$ the polynomials are respectively powers of a_1, a_2, $a_1^2+2a_3$ while for $n>3$ there are no such polynomials.

REMARK. An unfortunate misprint in formula (37) [6, p. 427] should be pointed out, namely the whole second line of the right-hand side should be conjugated.

3. We have observed that the bounds for the polynomials obtained by considering the family S have been found by elementary methods.

On the other hand the bound for $b_1^2+2b_3$ has so far been given only by Springer, using the relatively deep variational method. We want now to point out that the bounds for $a_1^2+2a_3$ and $b_1^2+2b_3$ can also be found in an elementary fashion.

We notice first that if $f\in\Sigma$, $\phi\in\Sigma^{-1}$ are inverse to one another and have expansions as at the beginning of §2, then $b_1=-a_1$, $b_2=-a_2$, $b_3=-a_3-a_1^2$. In particular $2b_3+b_1^2=-(2a_3+a_1^2)$. Thus it is enough to find the precise bound for either of the above polynomials.

Now let $f(z)=\mathrm{Re}^{i\Phi}$. Let C be the image of $|z|=r$ by f. Then it is well known that $\int_C R^4 d\Phi>0$ (see for example [2] or [1]). Evaluating the integral we find

$$r^4-\frac{2|a_2|^2}{r^2}-\frac{|2a_3+a_1^2|^2}{r^4}-\cdots>0.$$

Thus $|2a_3+a_1^2|\leqq 1$ and this is the precise bound, attained for $f(z)=z+e^{i\alpha}/z$ (α real).

By the above remark we have also $|b_1^2+2b_3|\leqq 1$. As Springer observed, this implies $|b_3|\leqq 1$ and this is the precise bound, attained for the inverse of $f(z)=z+e^{i\alpha}/z$ (α real).

We now remark that in all cases where the differential equation arising from the extremum of a w-homogeneous polynomial has just one term involving the extremal function, so that integration is easily performed, the bound for the polynomial can already be obtained by elementary methods.

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