# CONCERNING CARTAN'S CRITERION

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1. **Introduction.** Let $\mathfrak{L}$ be a Lie algebra over a field $\mathfrak{F}$ and let $\mathfrak{L}^{(0)}$, $\mathfrak{L}'$, $\mathfrak{L}''$, $\cdots$ be the *derived sequence* of $\mathfrak{L}$ where $\mathfrak{L}^{(0)}=\mathfrak{L}$ and $\mathfrak{L}^{(i+1)}=\mathfrak{L}^{(i)}\mathfrak{L}^{(i)}$. Let $t(R_x)$ denote the trace of a right multiplication $R_x$ of an element $x$ of $\mathfrak{L}$. Without further mention we shall assume that the base fields of all Lie and associative algebras considered have characteristic zero.

Cartan's criterion for solvability states that if $t(R_x^2)=0$ for all $x$ of $\mathfrak{L}^{(i)}$ for some $i\geqq 0$, then $\mathfrak{L}$ is solvable. On the other hand if $\mathfrak{L}$ is solvable then $t(R_x^2)=0$ for all $x$ of $\mathfrak{L}^{(i)}$ for $i\geqq 1$, but it is not true in general for $i=0$. However we shall show that if $\mathfrak{L}$ is solvable it can be imbedded in a solvable Lie algebra $\mathfrak{A}$ over an algebraic extension $\mathfrak{K}$ of $\mathfrak{F}$ such that $\mathfrak{A}$ has this property.

But we shall prove a much more general result, namely

THEOREM 1. *Let $\mathfrak{L}=\mathfrak{S}+\mathfrak{N}$ be the Levi-Whitehead decomposition of a Lie algebra $\mathfrak{L}$ over a field $\mathfrak{F}$, where $\mathfrak{N}$ is the radical and $\mathfrak{S}$ is either zero or semi-simple. Then there is a Lie algebra $\mathfrak{A}=\mathfrak{S}_{\mathfrak{K}}+\mathfrak{R}$ over an algebraic extension $\mathfrak{K}$ of $\mathfrak{F}$ whose radical $\mathfrak{R}$ contains $\mathfrak{N}$ and is the set of all $x$ of $\mathfrak{A}$ such that $t(R_xR_y)=0$ for all $y$ of $\mathfrak{A}$.*

The radical of a Lie algebra $\mathfrak{L}$ is [2, p. 14] the set of all $x$ of $\mathfrak{L}$ such that $t(R_xR_y)=0$ for all $y$ of $\mathfrak{L}'$ but the radical is not always the set of all $x$ of $\mathfrak{L}$ such that $t(R_xR_y)=0$ for every $y$ of $\mathfrak{L}$ as is the case for an associative algebra. Theorem 1 is an analogue to the associative case. It is also true that every associative algebra over $\mathfrak{F}$ has a faithful representation $x\rightarrow Q_x$ by matrices whose elements are in $\mathfrak{F}$, such that the radical is the set of all $x$ such that $t(Q_xQ_y)=0$ for every $y$. The corresponding statement for Lie algebras is not true, but we do obtain the following theorem.

THEOREM 2. *Every Lie algebra $\mathfrak{L}$ over a field $\mathfrak{F}$ has a faithful representation $x\rightarrow Q_x$ whose matrices have elements in an algebraic extension $\mathfrak{K}$ of $\mathfrak{F}$, such that the radical $\mathfrak{N}$ of $\mathfrak{L}$ is the set of all $x$ of $\mathfrak{L}$ such that $t(Q_xQ_y)=0$ for every $y$ of $\mathfrak{L}$.*

2. **Four lemmas.** In order to prove these theorems we first prove four lemmas. Throughout this section we let $\mathfrak{L}$ be a Lie algebra with radical $\mathfrak{N}$ over a field $\mathfrak{F}$ and let $x\rightarrow S_x$ be any representation of $\mathfrak{L}$.

Presented to the Society, September 3, 1952, under the title *A converse of Cartan's criterion for solvability*; received by the editors November 9, 1952.

Our first lemma, where $x \to S_x$ is the adjoint representation, can be found in [**2**, p. 14].

LEMMA 1. *$t(S_xS_y)=0$ for all $x \in \mathfrak{N}$ and all $y \in \mathfrak{L}'$.*

If $\mathfrak{B}$ is a subalgebra of $\mathfrak{L}$ we let $\overline{\mathfrak{B}}$ be the Lie algebra of linear transformations consisting of the $S_x$ for all $x \in \mathfrak{B}$. Then $\overline{\mathfrak{N}}$ is the radical of $\overline{\mathfrak{L}}$ and hence [**3**, p. 106] $[\overline{\mathfrak{L}}\overline{\mathfrak{N}}] \subseteq \mathfrak{R}$ where $\mathfrak{R}$ is the radical of the enveloping associative algebra $\mathfrak{L}^*$ of $\overline{\mathfrak{L}}$. Consequently if $x \in \mathfrak{N}$ and $y=uv$ for $u, v \in \mathfrak{L}$ then $t(S_xS_y) =t(S_{xu}S_v)$ where $S_{xu}=[S_xS_u] \in \mathfrak{R}$ and so $t(S_xS_y)=0$.

LEMMA 2. *If $t(S_xS_y)=0$ for all $x, y$ of $\mathfrak{N}$, then $t(S_xS_y)=0$ for all $x$ of $\mathfrak{N}$ and all $y$ of $\mathfrak{L}$.*

Let $\mathfrak{L}=\mathfrak{S}+\mathfrak{N}$ be the Levi-Whitehead decomposition of $\mathfrak{L}$, where $\mathfrak{S}$ is semi-simple or is zero. Then if $y \in \mathfrak{L}$, $y=u+v$ where $u \in \mathfrak{S}$ and $v \in \mathfrak{N}$. Now by Lemma 1, if $x \in \mathfrak{N}$, $t(S_xS_u)=0$, since $\mathfrak{S}'=\mathfrak{S}$. Hence $t(S_xS_y)=t(S_xS_u)+t(S_xS_v)=0$.

LEMMA 3. *The set of equations*

$$\sum_{r=1}^{t} d_{pr}d_{qr} = a_{pq} \qquad (p \leqq q = 1, 2, \cdots, t)$$

*where $a_{pq} \in \mathfrak{F}$ have a solution for the $d_{ij}$ in an algebraic extension $\mathfrak{K}$ of $\mathfrak{F}$.*

Take $d_{ij}=0$ for $i<j$ and then the equations can be put in $t$ sets

$$\begin{aligned} &d_{11}d_{q1} &&= a_{1q}, \\ &d_{21}d_{q1} + d_{22}d_{q2} &&= a_{2q}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots &&, \\ &d_{q1}d_{q1} + d_{q2}d_{q2} + \cdots + d_{qq}d_{qq} &&= a_{qq} \qquad (q = 1, 2, \cdots, t). \end{aligned}$$

Evidently these can be solved successively to get a solution in an algebraic extension $\mathfrak{K}$ of $\mathfrak{F}$.

LEMMA 4. *$\mathfrak{L}$ has an equivalent representation $x \to Q_x$ whose matrices have elements in an algebraic extension $\mathfrak{K}$ of $\mathfrak{F}$, such that $t(Q_xQ_y)=0$ for all $x$ of $\mathfrak{N}$ and all $y$ of $\mathfrak{L}$.*

Let $e_1, e_2, \cdots, e_n$ be a basis for $\mathfrak{L}$ where $e_1, e_2, \cdots, e_k$ is a basis for $\mathfrak{N}$ and $e_{t+1}, \cdots, e_k$ is a basis for the set $\mathfrak{T}$ of all $x$ of $\mathfrak{N}$ such that $t(S_xS_y)=0$ for every $y$ of $\mathfrak{N}$. If $x=\sum x_ie_i$ let $D_x=\sum x_iD_i$ where $D_i$ is the diagonal matrix diag $[d_{i1}, d_{i2}, \cdots, d_{it}]$ if $i \leqq t$ and $D_i=0$ otherwise. If

$$Q_x = \begin{bmatrix} S_x & 0 \\ 0 & D_x \end{bmatrix},$$

then $D_{xy}=0$ for all $x, y \in \mathfrak{L}$ and hence $Q$ is a representation equivalent to $\mathfrak{S}$ in the sense that the correspondence $S_x \to Q_x$ is an isomorphism under addition and the commutator multiplication $[S_x S_y] = S_{xy} = S_x S_y - S_y S_x$. By Lemma 3 we can take the $d_{ij}$ in an algebraic extension $\mathfrak{K}$ of $\mathfrak{F}$ such that $t(Q_x Q_y)=0$ for all $x, y \in \mathfrak{N}$. Hence by Lemma 2 we have $t(Q_x Q_y)=0$ for all $x$ of $\mathfrak{N}$ and all $y$ of $\mathfrak{L}$.

3. **Proof of Theorem** 1. The set of all $x$ of $\mathfrak{N}$ such that $t(\overline{R}_x \overline{R}_y)=0$ for every $y$ of $\mathfrak{N}$, where $x \to \overline{R}_x$ is the adjoint representation of $\mathfrak{L}$, is an ideal $\mathfrak{T}$ of $\mathfrak{N}$. Let $e_1, e_2, \cdots, e_k$ be a basis for $\mathfrak{N}$ and $e_{k+1}, \cdots, e_n$ be a basis for $\mathfrak{S}$ with $e_{t+1}, \cdots, e_k$ a basis for $\mathfrak{T}$. We construct a non-associative algebra $\mathfrak{A}$ over a field $\mathfrak{K}=\mathfrak{F}(d_{ij})$ by adjoining the additional basal elements $w_1, w_2, \cdots, w_t$ to $\mathfrak{L}$ and defining

$$w_j e_i = -e_i w_j = d_{ij} w_j \qquad (i, j = 1, 2, \cdots, t),$$

the products of the $w$'s with the other $e$'s and with themselves being defined to be zero.

It follows that $\mathfrak{A}$ is a Lie algebra, for by Lemma 1 and our construction,

$$(e_i e_j) w_k = 0 \qquad (i, j = 1, 2, \cdots, n;\ k = 1, 2, \cdots, t),$$

while on the other hand

$$e_i(e_j w_k) + (e_i w_k) e_j = 0 \qquad (i, j = 1, 2, \cdots, n;\ k = 1, 2, \cdots, t).$$

The ideal $\mathfrak{R}$ with basis $e_1, e_2, \cdots, e_k, w_1, w_2, \cdots, w_t$ is the radical of $\mathfrak{A}$.

Now if $x \to R_x$ is the adjoint representation of $\mathfrak{A}$ we have

$$t(R_{e_i} R_{e_j}) = t(\overline{R}_{e_i} \overline{R}_{e_j}) + \sum_{r=1}^{t} d_{ir} d_{jr} \qquad (i, j = 1, 2, \cdots, t).$$

But by Lemma 3 we can take the $d_{ij}$ in an algebraic extension of $\mathfrak{F}$ so that $t(R_{e_i} R_{e_j})=0$ for $(i, j=1, 2, \cdots, t)$ and then by Lemma 2 and the form of the $R_w$'s, with this choice of $\mathfrak{K}$, we have $t(R_x R_y)=0$ for all $x$ of $\mathfrak{R}$ and all $y$ of $\mathfrak{A}$.

The set of all $x$ of $\mathfrak{A}$ such that $t(R_x R_y)=0$ for every $y$ of $\mathfrak{A}$ is an ideal $\mathfrak{B}$ of $\mathfrak{A}$, and we have just shown that $\mathfrak{R} \subseteq \mathfrak{B}$. But if $x \to R_x'$ is the adjoint representation of $\mathfrak{B}$ we have $t(R_x' R_y') = t(R_x R_y)=0$ since $\mathfrak{B}$ is an ideal and hence by Cartan's criterion $\mathfrak{B}$ is solvable. Thus $\mathfrak{R}=\mathfrak{B}$ and the proof is complete.

4. **Proof of Theorem** 2. Harish-Chandra [1] and others have proved that every Lie algebra over a field of characteristic zero has a faithful representation. Consequently by Lemma 4, $\mathfrak{L}$ has a faithful representation $x \rightarrow Q_x$ whose matrices have elements in an algebraic extension $\mathfrak{K}$ of $\mathfrak{F}$ such that $t(Q_x Q_y) = 0$ for all $x$ of $\mathfrak{N}$ and all $y$ of $\mathfrak{L}$. We now apply another form of Cartan's criterion for solvability which states that if $t(A^2) = 0$ for all $A$ in a Lie algebra $\mathfrak{A}$ of linear transformations, than $\mathfrak{A}$ is solvable, and deduce that the ideal $\mathfrak{B}$ of all $x$ of $\mathfrak{L}$ such that $t(Q_x Q_y) = 0$ for every $y$ of $\mathfrak{L}$ is solvable. This proves the theorem for we now have $\mathfrak{B} = \mathfrak{N}$ as above.

## References

1. Harish-Chandra, *Faithful representations of Lie algebras*, Ann. of Math. vol. 50 (1949) pp. 68–76.

2. ———, *On the radical of a Lie algebra*, Proceedings of the American Mathematical Society vol. 1 (1950) pp. 14–17.

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# A SUBDIRECT-UNION REPRESENTATION FOR COMPLETELY DISTRIBUTIVE COMPLETE LATTICES

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1. **Introduction.** In [1],[1] Garrett Birkhoff makes the following remark: "Tarski has shown that any complete, completely distributive Boolean algebra is isomorphic with the field of subsets of a suitable set. One can also show that any closed sublattice of a direct union of complete chains is a complete, completely distributive lattice. The question is (no. 69), are there any other complete, completely distributive lattices?" This paper will answer Birkhoff's question by proving the following theorem:[2]

Theorem A. *Every completely distributive complete lattice is iso-*

Presented to the Society, October 25, 1952; received by the editors November 11, 1952.

[1] Numbers in brackets refer to the references cited at the end of the paper.

[2] Definitions and notations used here conform with those of [3], on which this paper is based.