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AN EXPRESSION FOR $\vartheta_1(nz)/\vartheta_1(z)$

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1. **Introduction.** In a recent paper [1] I used the formula

$$
\begin{aligned}
(1.1)\qquad & \prod \left[(1+q^n/z)(1+q^{n-1}z)(1-q^{2n-1}/z^2)(1-q^{2n-1}z^2)(1-q^n)\right] \\
&= \prod \left[(1-q^{3n-2}z^3)(1-q^{3n-1}/z^3)(1-q^{3n})\right] \\
&\quad + z\prod \left[(1-q^{3n-1}z^3)(1-q^{3n-2}/z^3)(1-q^{3n})\right],
\end{aligned}
$$

where, in the products, n takes all values from 1 to ∞, to simplify certain identities of the Rogers-Ramanujan type. It has been shown by Sears [2] (and independently by Miss Slater) that (1.1) can be derived from the relation connecting three products of four sigma functions, or alternatively from the corresponding relation connecting theta functions. Now (1.1) can be written in the form

$$
\begin{aligned}
(1.2)\qquad & \prod \frac{(1-q^{n-1}z^2)(1-q^n/z^2)}{(1-q^{n-1}z)(1-q^n/z)} \\
&= \prod \frac{(1-q^{3n})}{(1-q^n)} \times \Big[\prod (1-q^{3n-1}/z^3)(1-q^{3n-2}z^3) \\
&\quad + z\prod (1-q^{3n-2}/z^3)(1-q^{3n-1}z^3)\Big]
\end{aligned}
$$

and if we write

$$S(z;p) = \prod (1-p^{n-1}z)(1-p^n/z)$$

this formula can be written as

$$(1.3)\qquad \frac{S(z^2;q)}{S(z;q)} = \prod \frac{(1-q^{3n})}{(1-q^n)} \left[S(qz^3;q^3) + zS(q^2z^3;q^3)\right].$$

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But it is well known that [3, 2, 4]

$$(1.4) \qquad \sigma(x) = A \exp\,(Bx^2 - Cx)S(e^{2Cx};\, q^2),$$

where A, B, C are constants depending on ω_1, ω_2, and so (1.3) gives an expression for $\sigma(2z)/\sigma(z)$ as a sum. Alternatively we have, with the notation of Tannery and Molk,

$$(1.5) \qquad \vartheta_1(z) = iq^{1/4}e^{-i\pi z}\prod (1 - q^{2n})S(e^{2i\pi z};\, q^2),$$

and so (1.3) can be written in the form

$$(1.6) \quad \frac{\vartheta_1(2z)}{\vartheta_1(z)} = \frac{1}{\vartheta_1(\tau \mid 3\tau)}\left[e^{2i\pi z}\vartheta_1(3z + \tau \mid 3\tau) - e^{-2i\pi z}\vartheta_1(3z - \tau \mid 3\tau)\right].$$

It is well known [3, 2, 148] that $\vartheta_1(nz)/\vartheta_1(z)$ can be expressed as a single product of $(n^2 - 1)$ theta functions, but I have not been able to find any reference in the literature to an expression for $\vartheta_1(nz)/\vartheta_1(z)$ as a sum of n products, each product containing only $(n-1)$ theta functions.[1] The aim of this paper is to give such an expression.

2. **Notation.** With the above definition of $S(x;\, p)$, we write $S(x)$ for $S(x;\, p)$ when there is no ambiguity, and

$$S\begin{bmatrix} x_1, & x_2, & \cdots, & x_n; \\ y_1, & y_2, & \cdots, & y_m \end{bmatrix}$$

for

$$\frac{S(x_1)S(x_2)\cdots S(x_n)}{S(y_1)S(y_2)\cdots S(y_m)}.$$

3. **A theorem on products.** By considering the residues of the elliptic function

$$\frac{\sigma(u - b_1)\sigma(u - b_2)\cdots\sigma(u - b_n)}{\sigma(u - a_1)\sigma(u - a_2)\cdots\sigma(u - a_n)},$$

where $a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n$, it is seen that [3, 3, 46]

$$(3.1) \qquad \sum_{r=1}^{n} \frac{\sigma(a_r - b_1)\sigma(a_r - b_2)\cdots\sigma(a_r - b_n)}{\sigma(a_r - a_1)\sigma(a_r - a_2)\cdots {}^*\sigma(a_r - a_n)} = 0,$$

the star denoting that the vanishing factor $\sigma(a_r - a_r)$ has to be omitted. Using (1.4) and writing p for q^2, we see that (3.1) can be written in the form

[1] Tannery and Molk give an expression for $\vartheta_1(nz \mid n\tau)/\vartheta_1(z \mid \tau)$ as a single product of $(n-1)$ theta functions, but here we are considering $\vartheta_1(nz \mid \tau)/\vartheta_1(z \mid \tau)$.

$$\sum_{r=1}^{n} S\begin{bmatrix} A_r/B_1,\ A_r/B_2,\ \cdots,\ A_r/B_n; \\ A_r/A_1,\ \cdots *,\ A_r/A_n \end{bmatrix} = 0, \tag{3.2}$$

provided that $A_1A_2 \cdots A_n = B_1B_2 \cdots B_n$.

4. The expression for $\vartheta_1(nz)/\vartheta_1(z)$. In (3.2) take

$$\begin{aligned} B_2 &= B_1/p^{1/n},\ B_3 = B_1/p^{2/n},\ \cdots,\ B_n = B_1/p^{(n-1)/n}, \\ A_1 &= B_1z,\quad A_2 = B_1z/p^{1/n},\ \cdots,\ A_{n-1} = B_1z/p^{(n-2)/n}, \end{aligned}$$

so that $A_n = B_1/z^{n-1}p^{(n-1)/n}$. We then get

$$\sum_{=1}^{n-1} S\begin{bmatrix} z/p^{(r-1)/n},\ z/p^{(r-2)/n},\ \cdots,\ zp^{(n-r)/n}; \\ 1/p^{(r-1)/n},\ 1/p^{(r-2)/n},\ \cdots *,\ p^{(n-r-1)/n},\ z^np^{(n-r)/n} \end{bmatrix}$$
$$+ S\begin{bmatrix} 1/z^{n-1}p^{(n-1)/n},\ 1/z^{n-1}p^{(n-2)/n},\ \cdots,\ 1/z^{n-1}; \\ 1/z^np^{(n-1)/n},\ 1/z^np^{(n-2)/n},\ \cdots,\ 1/z^np^{1/n} \end{bmatrix} = 0.$$

But $S(1/x) = -(1/x)S(x)$, $S(x/p) = -(x/p)S(x)$, and so, changing r into $(n-r)$, we find that

$$\sum_{r=1}^{n-1} z^{n-r-1} S\begin{bmatrix} z,\ zp^{1/n},\ zp^{2/n},\ \cdots,\ zp^{(n-1)/n}; \\ p^{1/n},\ p^{2/n},\ \cdots *,\ p^{(n-1)/n},\ z^np^{r/n} \end{bmatrix}$$
$$= S\begin{bmatrix} z^{n-1},\ z^{n-1}p^{1/n},\ \cdots,\ z^{n-1}p^{(n-1)/n}; \\ z^np^{1/n},\ z^np^{2/n},\ \cdots,\ z^np^{(n-1)/n} \end{bmatrix},$$

where the star on the left-hand side denotes that $p^{r/n}$ is omitted. We now take $p = q^n$ and use the fact that

$$S(z;\ q) = S(z;\ q^n)S(zq;\ q^n) \cdots S(zq^{n-1};\ q^n),$$

and we obtain

$$\begin{aligned} \frac{S(z^{n-1};\ q)}{S(z;\ q)} &= S\begin{bmatrix} z^nq,\ z^nq^2,\ \cdots,\ z^nq^{n-1}; & q^n \\ q,\ q^2,\ \cdots,\ q^{n-1} & \end{bmatrix} \\ &\times \sum_{r=1}^{n-1} z^{n-r-1} S\begin{bmatrix} q^r; & q^n \\ z^nq^r & \end{bmatrix}. \end{aligned} \tag{4.1}$$

Using (1.5), this gives

$$\begin{aligned} \frac{\vartheta_1\{(n-1)z\}}{\vartheta_1\{(z)\}} &= \prod_{s=1}^{n-1} \frac{\vartheta_1(nz + s\tau \mid n\tau)}{\vartheta_1(s\tau \mid n\tau)} \\ &\times \sum_{r=1}^{n-1} \frac{\vartheta_1(r\tau \mid n\tau)}{\vartheta_1(nz + r\tau \mid n\tau)} e^{(n^2-n-2r)i\pi z}. \end{aligned} \tag{4.2}$$

If we change n into $(n+1)$, we see that we have expressed $\vartheta_1(nz)/\vartheta_1(z)$ as the sum of n products, each product containing $(n-1)$ theta functions. The case $n=3$ in (4.2) is equivalent to (1.6).

From (4.2) we can derive corresponding formulae for $\vartheta_2(nz)/\vartheta_2(z)$, $\vartheta_3(nz)/\vartheta_3(z)$, and $\vartheta_4(nz)/\vartheta_4(z)$ when n is an odd integer by changing z into $z+1/2$, $z+\tau/2$, $z+1/2+\tau/2$. We can also obtain formulae for $\vartheta_1(nz)/\vartheta_2(z)$, $\vartheta_1(nz)/\vartheta_3(z)$, and $\vartheta_1(nz)/\vartheta_4(z)$ when n is even by changing z in the same way.

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