4. **Proof of Theorem** 2. Harish-Chandra [1] and others have proved that every Lie algebra over a field of characteristic zero has a faithful representation. Consequently by Lemma 4, $\mathfrak{L}$ has a faithful representation $x \to Q_x$ whose matrices have elements in an algebraic extension $\mathfrak{K}$ of $\mathfrak{F}$ such that $t(Q_xQ_y)=0$ for all x of $\mathfrak{N}$ and all y of $\mathfrak{L}$. We now apply another form of Cartan's criterion for solvability which states that if $t(A^2)=0$ for all A in a Lie algebra $\mathfrak{A}$ of linear transformations, than $\mathfrak{A}$ is solvable, and deduce that the ideal $\mathfrak{B}$ of all x of $\mathfrak{L}$ such that $t(Q_xQ_y)=0$ for every y of $\mathfrak{L}$ is solvable. This proves the theorem for we now have $\mathfrak{B}=\mathfrak{N}$ as above.

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A SUBDIRECT-UNION REPRESENTATION FOR COMPLETELY DISTRIBUTIVE COMPLETE LATTICES

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1. **Introduction.** In [1],[1] Garrett Birkhoff makes the following remark: "Tarski has shown that any complete, completely distributive Boolean algebra is isomorphic with the field of subsets of a suitable set. One can also show that any closed sublattice of a direct union of complete chains is a complete, completely distributive lattice. The question is (no. 69), are there any other complete, completely distributive lattices?" This paper will answer Birkhoff's question by proving the following theorem:[2]

Theorem A. *Every completely distributive complete lattice is iso-*

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[1] Numbers in brackets refer to the references cited at the end of the paper.

[2] Definitions and notations used here conform with those of [3], on which this paper is based.

morphic with a closed sublattice of the direct union of a family of complete chains.

2. A characterization of complete distributivity.

Definition 1. If L is a partially ordered set and if M is a subset of L such that if $x \in M$ and $y \leqq x$, then $y \in M$, then M is called a *semi-ideal of L*. Let $R(L)$ denote the complete lattice of semi-ideals of L.

Definition 2. If L is a complete lattice and $x \in L$, then let $K(x) = \prod\{M \mid M \in R(L) \text{ and } x \leqq \bigcup M\}$.

Lemma 1. *If L is a complete lattice, then*

(A) *if $x \in L$, then $\bigcup K(x) \leqq x$;*

(B) *if $x \in L$, $y \in L$, and $x \leqq y$, then $K(x) \subset K(y)$;*

(C) *if $A \subset L$, then $\sum\{K(a) \mid a \in A\} = K(\bigcup A)$.*

Proof. If $x \in L$, then $\{t \mid t \leqq x\} \in R(L)$ and $\bigcup\{t \mid t \leqq x\} = x$, so that $K(x) \subset \{t \mid t \leqq x\}$. Therefore $\bigcup K(x) \leqq x$. If $x \in L$, $y \in L$, and $x \leqq y$, then $\{M \mid M \in R(L) \text{ and } y \leqq \bigcup M\} \subset \{M \mid M \in R(L) \text{ and } x \leqq \bigcup M\}$; hence $K(x) \subset K(y)$. If $A \subset L$ and $t \notin \sum\{K(a) \mid a \in A\}$, then for every $a \in A$, $t \notin K(a)$ and one can choose an $M_a \in R(L)$ such that $t \notin M_a$ and $a \leqq \bigcup M_a$. Then $t \notin \sum\{M_a \mid a \in A\}$. Moreover, $K(\bigcup A) \subset \sum\{M_a \mid a \in A\}$, since $\sum\{M_a \mid a \in A\} \in R(L)$ and $\bigcup A \leqq \bigcup\{\bigcup M_a \mid a \in A\} = \bigcup \sum\{M_a \mid a \in A\}$. Hence $t \notin K(\bigcup A)$. On the other hand, if $t \in \sum\{K(a) \mid a \in A\}$, then $t \in K(a)$ for some $a \in A$, and since $a \leqq \bigcup A$, $t \in K(\bigcup A)$. Therefore $\sum\{K(a) \mid a \in A\} = K(\bigcup A)$.

Lemma 2. *In order that a complete lattice L be completely distributive it is necessary and sufficient that if $\{M_\gamma \mid \gamma \in C\}$ is a family of semi-ideals of L, then $\bigcap\{\bigcup M_\gamma \mid \gamma \in C\} \leqq \bigcup \prod\{M_\gamma \mid \gamma \in C\}$.*

This follows from Theorem 1 and Lemma 5 of [3].

Theorem 1. *In order that a complete lattice L be completely distributive it is necessary and sufficient that for every $x \in L$, $\bigcup K(x) = x$.*

Proof. To prove necessity, let the complete lattice L be completely distributive. If $x \in L$, then $x \leqq \bigcap\{\bigcup M \mid M \in R(L) \text{ and } x \leqq \bigcup M\} \leqq \bigcup \prod\{M \mid M \in R(L) \text{ and } x \leqq \bigcup M\} = \bigcup K(x)$. This, together with Lemma 1(A), implies that $\bigcup K(x) = x$ for every $x \in L$.

To prove sufficiency, let L be a complete lattice such that for every $x \in L$, $\bigcup K(x) = x$. If $\{M_\gamma \mid \gamma \in C\}$ is a family of semi-ideals of L and if $t \in K(\bigcap\{\bigcup M_\gamma \mid \gamma \in C\})$, then for every $\gamma \in C$, $t \in K(\bigcup M_\gamma)$, by Lemma 1(B), and $t \in \sum\{K(x) \mid x \in M_\gamma\}$, by Lemma 1(C). For every $\gamma \in C$ one can choose an $x_\gamma \in M_\gamma$ such that $t \in K(x_\gamma)$ and then $t \leqq \bigcup K(x_\gamma) = x_\gamma$. Hence $t \leqq \bigcap\{x_\gamma \mid \gamma \in C\}$ and, since $\bigcap\{x_\gamma \mid \gamma \in C\} \in \prod\{M_\gamma \mid \gamma \in C\}$, $t \in \prod\{M_\gamma \mid \gamma \in C\}$. Therefore $K(\bigcap\{\bigcup M_\gamma \mid \gamma \in C\})$

$\subset \prod\{M_\gamma | \gamma \in C\}$. It follows that $\bigcap\{\bigcup M_\gamma | \gamma \in C\} = \bigcup K(\bigcap\{\bigcup M_\gamma | \gamma \in C\}) \leqq \bigcup \prod\{M_\gamma | \gamma \in C\}$, and, by Lemma 2, L is completely distributive.

DEFINITION 3. If L is a complete lattice, let ρ be the binary relation on L defined as follows: $x\rho y$ if and only if $x \in L$, $y \in L$, and $x \in K(y)$.

DEFINITION 4. If σ is a binary relation on a set X, let $\sigma \circ \sigma$ be the binary relation on X defined as follows: $x\sigma \circ \sigma y$ if and only if there exists a z such that $x\sigma z$ and $z\sigma y$.

COROLLARY. *If L is a completely distributive complete lattice, then $\rho = \rho \circ \rho$.*

PROOF. For every $x \in L$, $K(x) = K(\bigcup K(x)) = \sum\{K(a) | a \in K(x)\}$, by Theorem 1 and Lemma 1(C). It then follows that $\rho = \rho \circ \rho$.

The nonmodular lattice of five elements is a complete lattice in which $\rho = \rho \circ \rho$ and which is not completely distributive. Hence the converse of the corollary is not true.

3. **Relations $\sigma = \sigma \circ \sigma$.** Let X be a set and let σ be a binary relation on X such that $\sigma = \sigma \circ \sigma$.

DEFINITION 5. If $A \subset X$, let $\phi(A)$ be the set of $x \in X$ such that there exists a $y \in A$ such that $x\sigma y$. Let $L(\sigma)$ be the family $\{\phi(A) | A \subset X\}$, partially ordered by set-inclusion.

THEOREM 2. *If σ is a binary relation on a set X and if $\sigma = \sigma \circ \sigma$, then $L(\sigma)$ is a completely distributive complete lattice. If, in addition, σ is reflexive, then $L(\sigma)$ is a complete ring of sets.*

PROOF. If $\{A_\gamma | \gamma \in C\}$ is a family of subsets of X, and if $x \in \phi(\sum\{A_\gamma | \gamma \in C\})$, then there is a $\gamma \in C$ and a $y \in A_\gamma$ such that $x\sigma y$. Then $x \in \phi(A_\gamma)$; hence $x \in \sum\{\phi(A_\gamma) | \gamma \in C\}$. This proves that $\phi(\sum\{A_\gamma | \gamma \in C\}) \subset \sum\{\phi(A_\gamma) | \gamma \in C\}$. For every $\gamma \in C$, $\phi(A_\gamma) \subset \phi(\sum\{A_\gamma | \gamma \in C\})$. Therefore $\sum\{\phi(A_\gamma) | \gamma \in C\} = \phi(\sum\{A_\gamma | \gamma \in C\})$ and $L(\sigma)$ is closed with respect to union. Hence $L(\sigma)$ is a complete lattice, in which joins are unions; that is, $\bigcup\{\phi(A_\gamma) | \gamma \in C\} = \sum\{\phi(A_\gamma) | \gamma \in C\}$.

If $A \subset X$ and $x \in \phi(A)$, then there is a $y \in A$ such that $x\sigma y$. Since $\sigma = \sigma \circ \sigma$, there is a t such that $x\sigma t$ and $t\sigma y$. Hence if $x \in \phi(A)$, then there is a $t \in \phi(A)$ such that $x \in \phi(\{t\})$. Therefore $\phi(A) \subset \sum\{\phi(\{t\}) | t \in \phi(A)\}$.

If $t \in \phi(A)$ and M is a semi-ideal in $L(\sigma)$ such that $\phi(A) \subset \sum M$, then there exists a $B \subset X$ such that $\phi(B) \in M$ and $t \in \phi(B)$. Then $\phi(\{t\}) \subset \phi(B)$; hence $\phi(\{t\}) \in M$. Therefore, if $t \in \phi(A)$, then $\phi(\{t\}) \in K(\phi(A))$. It follows that $\phi(A) \subset \sum K(\phi(A)) = \bigcup K(\phi(A))$. This, together with Lemma 1(A), implies that $\phi(A) = \bigcup K(\phi(A))$ for every $A \subset X$. This proves that $L(\sigma)$ is completely distributive.

If, in addition, σ is reflexive, then for every $A \subset X$, $A \subset \phi(A)$. Hence if $\{A_\gamma | \gamma \in C\}$ is a family of subsets of X, then $\prod \{\phi(A_\gamma) | \gamma \in C\} \subset \phi(\prod \{\phi(A_\gamma) | \gamma \in C\})$. On the other hand, for every $\gamma \in C$, $\phi(\prod \{\phi(A_\gamma) | \gamma \in C\}) \subset \phi(\phi(A_\gamma)) = \phi(A_\gamma)$. Therefore, $\prod \{\phi(A_\gamma) | \gamma \in C\} = \phi(\prod \{\phi(A_\gamma) | \gamma \in C\})$, and $L(\sigma)$ is closed with respect to intersection as well as union. In other words, $L(\sigma)$ is a complete ring of sets.

If $\sigma = \sigma \circ \sigma$ and σ is reflexive, then σ is a quasi-ordering. Theorem 2 shows that the relation between completely distributive complete lattices and relations $\sigma = \sigma \circ \sigma$ is a generalization of the relation between complete rings of sets and quasi-orderings. The latter relation has been studied by G. Birkhoff in [2].

4. **Proof of Theorem** A.

Definition 6. If σ is a binary relation on a set X, and if C is a subset of X such that if $x \in C$ and $y \in C$, then either $x = y$ or $x\sigma y$ or $y\sigma x$, then C is called a *chain in* σ. If C is a chain in σ which is not properly contained in any chain in σ, then C is called a *maximal chain in* σ.

It follows from Zorn's Lemma that every chain in σ is contained in a maximal chain in σ.

Let L be a completely distributive complete lattice and let Γ be the family of maximal chains in ρ. If $C \in \Gamma$ and $a \in L$, let $f(C, a)$ be the set of $t \in C$ such that there exists an $x \in C$ such that $t\rho x\rho a$. If $C \in \Gamma$, let $F_C = \{f(C, a) | a \in L\}$.

For every $x \in L$, $\sum \{f(C, x) | C \in \Gamma\} = K(x)$. For if $t \in K(x)$, then $\{t, x\}$ is a chain in ρ, so that there is a $C \in \Gamma$ such that $\{t, x\} \subset C$. Since C is maximal and $\rho = \rho \circ \rho$, there is a $y \in C$ such that $t\rho y\rho x$. Then $t \in f(C, x)$. Therefore $K(x) \subset \sum \{f(C, x) | C \in \Gamma\}$. On the other hand, if $t \in f(C, x)$ for some $C \in \Gamma$, then $t \in K(x)$. Therefore $\sum \{f(C, x) | C \in \Gamma\} \subset K(x)$.

If $a \in L$ and $b \in L$, then either $f(C, a) \subset f(C, b)$ or $f(C, b) \subset f(C, a)$. For if $f(C, a)$ is not contained in $f(C, b)$, then there is a $t \in f(C, a)$ such that $t \notin f(C, b)$. If $y \in f(C, b)$, then neither $t = y$ nor $t\rho y$; otherwise $t \in f(C, b)$. Hence $y\rho t$ and $y \in f(C, a)$. Then $f(C, b) \subset f(C, a)$. Therefore, for every $C \in \Gamma$, F_C is a chain in the relation of set-inclusion on the set of subsets of C.

If $C \in \Gamma$ and $A \subset L$, then $\sum \{f(C, a) | a \in A\} = f(C, \bigcup A)$. For $t \in f(C, \bigcup A)$ if and only if $t \in C$ and there is an $x \in C$ such that $t\rho x\rho \bigcup A$. By Lemma 1(C), $t\rho x\rho \bigcup A$ if and only if there exists an $a \in A$ such that $t\rho x\rho a$. Hence $t \in f(C, \bigcup A)$ if and only if there exists an $a \in A$ such that $t \in f(C, a)$. Therefore F_C is closed with respect to union, for every

$C \in \Gamma$. It follows that for every $C \in \Gamma$, F_C is a complete chain in which, if $F \subset F_C$, $\bigcup F = \sum F$ and $\bigcap F = \sum \{f(C, b) \mid b \in L$ and $f(C, b) \subset \prod F\}$.

If $C \in \Gamma$ and $A \subset L$, then $\bigcap \{f(C, a) \mid a \in A\} = f(C, \bigcap A)$. For if $t \in \bigcap \{f(C, a) \mid a \in A\}$, then there exists a $b \in L$ such that $t \in f(C, b)$ and $f(C, b) \subset \prod \{f(C, a) \mid a \in A\}$. Then $t \in C$ and there exists an $s \in C$ such that $t \rho s \rho b$. Since $\rho = \rho \circ \rho$ and since C is a maximal chain in ρ, there exists $u \in C$ and $y \in C$ such that $t \rho u \rho y \rho s \rho b$. Then $y \in f(C, b)$, and for every $a \in A$, $y \in f(C, a)$. Hence $y \leqq a$ for every $a \in A$; so that $y \leqq \bigcap A$. By Lemma 1(B), $t \rho u \rho \bigcap A$, and $t \in f(C, \bigcap A)$. Therefore $\bigcap \{f(C, a) \mid a \in A\} \subset f(C, \bigcap A)$. On the other hand, $f(C, \bigcap A) \subset \prod \{f(C, a) \mid a \in A\}$, by Lemma 1(B). Therefore, $f(C, \bigcap A) \subset \bigcap \{f(C, a) \mid a \in A\}$.

Let D be the direct union of the family of complete chains $\{F_C \mid C \in \Gamma\}$. D consists of all functions θ: $\Gamma \to \sum \{F_C \mid C \in \Gamma\}$ such that for every $C \in \Gamma$, $\theta(C) \in F_C$. Furthermore D is a complete lattice in which, if $D_1 \subset D$, then $(\bigcup D_1)(C) = \bigcup \{\theta(C) \mid \theta \in D_1\}$ and $(\bigcap D_1)(C) = \bigcap \{\theta(C) \mid \theta \in D_1\}$, for every $C \in \Gamma$.

For every $a \in L$, let θ_a be the member of D such that for every $C \in \Gamma$, $\theta_a(C) = f(C, a)$. Let $L^* = \{\theta_a \mid a \in L\}$. The mapping $a \to \theta_a$ is a one-to-one mapping of L onto L^*. For if $\theta_a = \theta_b$, then for every $C \in \Gamma$, $f(C, a) = f(C, b)$. Then $a = \bigcup K(a) = \bigcup \sum \{f(C, a) \mid C \in \Gamma\} = \bigcup \sum \{f(C, b) \mid C \in \Gamma\} = \bigcup K(b) = b$.

If $A \subset L$, then $\bigcup \{\theta_a \mid a \in A\} = \theta_{\bigcup A}$. For, if $C \in \Gamma$, then $(\bigcup \{\theta_a \mid a \in A\})(C) = \bigcup \{\theta_a(C) \mid a \in A\} = \bigcup \{f(C, a) \mid a \in A\} = \sum \{f(C, a) \mid a \in A\} = f(C, \bigcup A) = \theta_{\bigcup A}(C)$. Also if $A \subset L$, then $\bigcap \{\theta_a \mid a \in A\} = \theta_{\bigcap A}$. For, if $C \in \Gamma$, then $(\bigcap \{\theta_a \mid a \in A\})(C) = \bigcap \{\theta_a(C) \mid a \in A\} = \bigcap \{f(C, a) \mid a \in A\} = f(C, \bigcap A) = \theta_{\bigcap A}(C)$. It follows that L^* is a closed sublattice of D, and that the mapping $a \to \theta_a$ is a complete-isomorphism of L onto L^*.

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