4. Proof of Theorem 2. Harish-Chandra [1] and others have proved that every Lie algebra over a field of characteristic zero has a faithful representation. Consequently by Lemma 4, $\mathfrak{g}$ has a faithful representation $x \rightarrow Q_x$ whose matrices have elements in an algebraic extension $\mathfrak{R}$ of $\mathfrak{g}$ such that $t(Q_x Q_y) = 0$ for all $x \in \mathfrak{R}$ and all $y \in \mathfrak{g}$. We now apply another form of Cartan's criterion for solvability which states that if $t(A^2) = 0$ for all $A$ in a Lie algebra $\mathfrak{A}$ of linear transformations, then $\mathfrak{A}$ is solvable, and deduce that the ideal $\mathfrak{B}$ of all $x$ of $\mathfrak{g}$ such that $t(Q_x Q_y) = 0$ for every $y \in \mathfrak{g}$ is solvable. This proves the theorem for we now have $\mathfrak{B} = \mathfrak{R}$ as above.

References


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A SUBDIRECT-UNION REPRESENTATION FOR COMPLETELY DISTRIBUTIVE COMPLETE LATTICES

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1. Introduction. In [1], Garrett Birkhoff makes the following remark: "Tarski has shown that any complete, completely distributive Boolean algebra is isomorphic with the field of subsets of a suitable set. One can also show that any closed sublattice of a direct union of complete chains is a complete, completely distributive lattice. The question is (no. 69), are there any other complete, completely distributive lattices?" This paper will answer Birkhoff's question by proving the following theorem:2

Theorem A. Every completely distributive complete lattice is iso-
morphic with a closed sublattice of the direct union of a family of complete chains.

2. A characterization of complete distributivity.

**Definition 1.** If $L$ is a partially ordered set and if $M$ is a subset of $L$ such that if $x \in M$ and $y \leq x$, then $y \in M$, then $M$ is called a semi-ideal of $L$. Let $R(L)$ denote the complete lattice of semi-ideals of $L$.

**Definition 2.** If $L$ is a complete lattice and $x \in L$, then let $K(x) = \prod \{ M \mid M \in R(L) \text{ and } x \leq \bigcup M \}$.

**Lemma 1.** If $L$ is a complete lattice, then
(A) if $x \in L$, then $\bigcup K(x) \leq x$;
(B) if $x \in L$, $y \in L$, and $x \leq y$, then $K(x) \subseteq K(y)$;
(C) if $A \subseteq L$, then $\bigcup \{ K(a) \mid a \in A \} = K(\bigcup A)$.

**Proof.** If $x \in L$, then $\{ t \mid t \leq x \} \in R(L)$ and $\bigcup \{ t \mid t \leq x \} = x$, so that $K(x) \subseteq \{ t \mid t \leq x \}$. Therefore $\bigcup K(x) \leq x$. If $x \in L$, $y \in L$, and $x \leq y$, then $\{ M \mid M \in R(L) \text{ and } y \leq \bigcup M \} \subseteq \{ M \mid M \in R(L) \text{ and } x \leq \bigcup M \}$; hence $K(x) \subseteq K(y)$. If $A \subseteq L$ and $t \in \bigcup \{ K(a) \mid a \in A \}$, then for every $a \in A$, $t \in K(a)$ and one can choose an $M_a \subseteq R(L)$ such that $t \in M_a$ and $a \leq \bigcup M_a$. Then $t \in \bigcup \{ K(a) \mid a \in A \}$. Moreover, $K(\bigcup A) \subseteq \bigcup \{ K(a) \mid a \in A \}$, since $\bigcup \{ K(a) \mid a \in A \} \subseteq R(L)$ and $\bigcup A \subseteq \bigcup \{ K(a) \mid a \in A \}$.

**Lemma 2.** In order that a complete lattice $L$ be completely distributive it is necessary and sufficient that if $\{ M_y \mid y \in C \}$ is a family of semi-ideals of $L$, then $\bigcap \{ \bigcup M_y \mid y \in C \} \leq \bigcup \{ M_y \mid y \in C \}$.

This follows from Theorem 1 and Lemma 5 of [3].

**Theorem 1.** In order that a complete lattice $L$ be completely distributive it is necessary and sufficient that for every $x \in L$, $\bigcup K(x) = x$.

**Proof.** To prove necessity, let the complete lattice $L$ be completely distributive. If $x \in L$, then $x \leq \bigcap \{ \bigcup M \mid M \in R(L) \text{ and } x \leq \bigcup M \} \leq \bigcup \{ M \mid M \in R(L) \text{ and } x \leq \bigcup M \} = \bigcup K(x)$. This, together with Lemma 1(A), implies that $\bigcup K(x) = x$ for every $x \in L$.

To prove sufficiency, let $L$ be a complete lattice such that for every $x \in L$, $\bigcup K(x) = x$. If $\{ M_y \mid y \in C \}$ is a family of semi-ideals of $L$ and if $t \in K(\bigcap \{ \bigcup M \mid M \in R(L) \text{ and } x \leq \bigcup M \})$, then for every $y \in C$, $t \in K(M_y)$, by Lemma 1(B), and $t \in \bigcup \{ K(x) \mid x \in M_y \}$, by Lemma 1(C). For every $y \in C$ one can choose an $x_y \in M_y$ such that $t \in K(x_y)$ and then $t \leq \bigcap K(x_y) = x_y$. Hence $t \leq \bigcap \{ x_y \mid y \in C \}$ and, since $\bigcap \{ x_y \mid y \in C \} \in \bigcap \{ M_y \mid y \in C \}$, $t \in \bigcap \{ M_y \mid y \in C \}$. Therefore $K(\bigcap \{ M_y \mid y \in C \})$
\[ \prod \{ M_{\gamma} | \gamma \in C \} \]. It follows that \( \bigcap \bigcup M_{\gamma} \bigcap \gamma \in C \) = \( \bigcap \bigcup M_{\gamma} \bigcap \gamma \in C \), and, by Lemma 2, \( L \) is completely distributive.

**Definition 3.** If \( L \) is a complete lattice, let \( \rho \) be the binary relation on \( L \) defined as follows: \( x \rho y \) if and only if \( x \in L, \gamma \in L, \) and \( x \in K(y) \).

**Definition 4.** If \( \sigma \) is a binary relation on a set \( X \), let \( \sigma \circ \sigma \) be the binary relation on \( X \) defined as follows: \( x \sigma \circ \sigma y \) if and only if there exists a \( z \) such that \( x \sigma z \) and \( z \sigma y \).

**Corollary.** If \( L \) is a completely distributive complete lattice, then \( \rho = \rho \circ \rho \).

**Proof.** For every \( x \in L, K(x) = K(\bigcup K(x)) = \bigcup \{ K(a) | a \in K(x) \} \), by Theorem 1 and Lemma 1(C). It then follows that \( \rho = \rho \circ \rho \).

The nonmodular lattice of five elements is a complete lattice in which \( \rho = \rho \circ \rho \) and which is not completely distributive. Hence the converse of the corollary is not true.

3. **Relations \( \sigma = \sigma \circ \sigma \).** Let \( X \) be a set and let \( \sigma \) be a binary relation on \( X \) such that \( \sigma = \sigma \circ \sigma \).

**Definition 5.** If \( A \subseteq X \), let \( \phi(A) \) be the set of \( x \in X \) such that there exists a \( y \in A \) such that \( x \sigma y \). Let \( L(\sigma) \) be the family \( \{ \phi(A) | A \subseteq X \} \), partially ordered by set-inclusion.

**Theorem 2.** If \( \sigma \) is a binary relation on a set \( X \) and if \( \sigma = \sigma \circ \sigma \), then \( L(\sigma) \) is a completely distributive complete lattice. If, in addition, \( \sigma \) is reflexive, then \( L(\sigma) \) is a complete ring of sets.

**Proof.** If \( \{ A_{\gamma} | \gamma \in C \} \) is a family of subsets of \( X \), and if \( x \in \phi(\bigcup \{ A_{\gamma} | \gamma \in C \}) \), then there is a \( \gamma \in C \) and a \( A_{\gamma} \subseteq X \) such that \( x \sigma y \). Then \( x \in \phi(A_{\gamma}) \); hence \( x \in \sum \{ \phi(A_{\gamma}) | \gamma \in C \} \). This proves that \( \phi(\bigcup \{ A_{\gamma} | \gamma \in C \}) \subseteq \sum \{ \phi(A_{\gamma}) | \gamma \in C \} \). For every \( \gamma \in C \), \( \phi(A) \subseteq \phi(\bigcup \{ A_{\gamma} | \gamma \in C \}) \). Therefore \( \sum \{ \phi(A_{\gamma}) | \gamma \in C \} = \phi(\bigcup \{ A_{\gamma} | \gamma \in C \}) \) and \( L(\sigma) \) is closed with respect to union. Hence \( L(\sigma) \) is a complete lattice, in which joins are unions; that is, \( \bigcup \{ \phi(A_{\gamma}) | \gamma \in C \} = \sum \{ \phi(A_{\gamma}) | \gamma \in C \} \).

If \( A \subseteq X \) and \( x \in \phi(A) \), then there is a \( y \in A \) such that \( x \sigma y \). Since \( \sigma = \sigma \circ \sigma \), there is a \( t \) such that \( x \sigma t \) and \( t \sigma y \). Hence if \( x \in \phi(A) \), then there is a \( t \in \phi(A) \) such that \( x \in \phi(\{ t \}) \). Therefore \( \phi(A) \subseteq \bigcup \{ \phi(\{ t \}) | t \in \phi(A) \} \).

If \( t \in \phi(A) \) and \( M \) is a semi-ideal in \( L(\sigma) \) such that \( \phi(A) \subseteq \bigcup M \), then there exists a \( B \subseteq X \) such that \( \phi(B) \subseteq M \) and \( t \in \phi(B) \). Then \( \phi(\{ t \}) \subseteq \phi(B) \); hence \( \phi(\{ t \}) \subseteq M \). Therefore, if \( t \in \phi(A) \), then \( \phi(\{ t \}) \subseteq K(\phi(A)) \). It follows that \( \phi(A) \subseteq \bigcup K(\phi(A)) = \bigcup K(\phi(A)) \). This,
together with Lemma 1(A), implies that $\phi(A) = \bigcup K(\phi(A))$ for every $A \subseteq X$. This proves that $L(\sigma)$ is completely distributive.

If, in addition, $\sigma$ is reflexive, then for every $A \subseteq X, A \subseteq \phi(A)$. Hence if $\{A_\gamma \mid \gamma \in C\}$ is a family of subsets of $X$, then $\prod \{\phi(A_\gamma) \mid \gamma \in C\} \subseteq \phi(\prod \{\phi(A_\gamma) \mid \gamma \in C\})$. On the other hand, for every $\gamma \in C$, $\phi(\prod \{\phi(A_\gamma) \mid \gamma \in C\}) \subset \phi(\phi(A_\gamma)) = \phi(A_\gamma)$. Therefore, $\prod \{\phi(A_\gamma) \mid \gamma \in C\} = \phi(\prod \{\phi(A_\gamma) \mid \gamma \in C\})$, and $L(\sigma)$ is closed with respect to intersection as well as union. In other words, $L(\sigma)$ is a complete ring of sets.

If $\sigma = \sigma \circ \sigma$ and $\sigma$ is reflexive, then $\sigma$ is a quasi-ordering. Theorem 2 shows that the relation between completely distributive complete lattices and relations $\sigma = \sigma \circ \sigma$ is a generalization of the relation between complete rings of sets and quasi-orderings. The latter relation has been studied by G. Birkhoff in [2].

4. Proof of Theorem A.

Definition 6. If $\sigma$ is a binary relation on a set $X$, and if $C$ is a subset of $X$ such that if $x \in C$ and $y \in C$, then either $x = y$ or $x \sigma y$ or $y \sigma x$, then $C$ is called a chain in $\sigma$. If $C$ is a chain in $\sigma$ which is not properly contained in any chain in $\sigma$, then $C$ is called a maximal chain in $\sigma$.

It follows from Zorn's Lemma that every chain in $\sigma$ is contained in a maximal chain in $\sigma$.

Let $L$ be a completely distributive complete lattice and let $\Gamma$ be the family of maximal chains in $\rho$. If $C \in \Gamma$ and $a \in L$, let $f(C, a)$ be the set of $t \in C$ such that there exists an $x \in C$ such that $t \rho x a$. If $C \in \Gamma$, let $F_C = \{f(C, a) \mid a \in L\}$.

For every $x \in L$, $\sum f(C, x) \mid C \in \Gamma = K(x)$. If $t \in K(x)$, then $\{t, x\}$ is a chain in $\rho$, so that there is a $C \in \Gamma$ such that $\{t, x\} \subseteq C$. Since $C$ is maximal and $\rho = \rho \circ \rho$, there is a $y \in C$ such that $t \rho y p x$. Then $t \in f(C, x)$. Therefore $K(x) \subseteq \sum f(C, x) \mid C \in \Gamma$. On the other hand, if $t \in f(C, x)$ for some $C \in \Gamma$, then $t \in K(x)$. Therefore $\sum f(C, x) \mid C \in \Gamma \subseteq K(x)$.

If $a \in L$ and $b \in L$, then either $f(C, a) \subseteq f(C, b)$ or $f(C, b) \subseteq f(C, a)$. For if $f(C, a)$ is not contained in $f(C, b)$, then there is a $t \in f(C, a)$ such that $t \notin f(C, b)$. If $y \in f(C, b)$, then neither $t = y$ nor $t \rho y$; otherwise $t \in f(C, b)$. Hence $y \notin f(C, a)$. Then $f(C, b) \subseteq f(C, a)$. Therefore, for every $C \in \Gamma$, $F_C$ is a chain in the relation of set-inclusion on the set of subsets of $C$.

If $C \in \Gamma$ and $A \subseteq L$, then $\sum f(C, a) \mid a \in A = f(C, U A)$. For $t \in f(C, U A)$ if and only if $t \in C$ and there is an $x \in C$ such that $t \rho x p A$. By Lemma 1(C), $t \rho x p U A$ if and only if there exists an $a \in A$ such that $t \rho x p a$. Hence $t \in f(C, U A)$ if and only if there exists an $a \in A$ such that $t \in f(C, a)$. Therefore $F_C$ is closed with respect to union, for every
If \( C \in \Gamma \), it follows that for every \( C \in \Gamma \), \( F_C \) is a complete chain in which, if \( F_C \subseteq F \), then \( \cap F = \{ f(C, b) \mid b \in L \} \) and \( \cup F = \{ f(C, b) \mid b \in L \} \).

If \( C \in \Gamma \) and \( A \subseteq L \), then \( \cap \{ f(C, a) \mid a \in A \} = f(C, \cap A) \). For if \( t \in \cap \{ f(C, a) \mid a \in A \} \), then there exists a \( b \in L \) such that \( t \in f(C, b) \) and \( f(C, b) \subseteq \prod \{ f(C, a) \mid a \in A \} \). Then \( t \in C \) and there exists an \( s \in C \) such that \( tpsb \). Since \( \rho = \rho \circ \rho \) and since \( C \) is a maximal chain in \( \rho \), there exists \( u \in C \) and \( y \in C \) such that \( tpuypsb \). Then \( y \in f(C, b) \), and for every \( a \in A \), \( y \in f(C, a) \). Hence \( y \leq a \) for every \( a \in A \); so that \( y \leq \cap A \). By Lemma 1(B), \( tpu p \cap A \), and \( t \in f(C, \cap A) \). Therefore \( \cap \{ f(C, a) \mid a \in A \} \subseteq f(C, \cap A) \). On the other hand, \( f(C, \cap A) \subseteq \prod \{ f(C, a) \mid a \in A \} \), by Lemma 1(B). Therefore, \( f(C, \cap A) \subseteq \cap \{ f(C, a) \mid a \in A \} \).

Let \( D \) be the direct union of the family of complete chains \( \{ F_C \mid C \in \Gamma \} \). \( D \) consists of all functions \( \theta : \Gamma \to \prod \{ f(C) \mid C \in \Gamma \} \) such that for every \( C \in \Gamma \), \( \theta(C) \subseteq F_C \). Furthermore \( D \) is a complete lattice in which, if \( D_1 \subseteq D \), then \( (\cup D_1)(C) = \cup \{ \theta(C) \mid \theta \in D_1 \} \) and \( (\cap D_1)(C) = \cap \{ \theta(C) \mid \theta \in D_1 \} \), for every \( C \in \Gamma \).

For every \( a \in L \), let \( \theta_a \) be the member of \( D \) such that for every \( C \in \Gamma \), \( \theta_a(C) = f(C, a) \). Let \( L^* = \{ \theta_a \mid a \in L \} \). The mapping \( a \rightarrow \theta_a \) is a one-to-one mapping of \( L \) onto \( L^* \). For if \( \theta_a = \theta_b \), then for every \( C \in \Gamma \), \( f(C, a) = f(C, b) \). Then \( a = \cup K(a) = \cup \{ f(C, a) \mid C \in \Gamma \} = \cup \sum \{ f(C, b) \mid C \in \Gamma \} = \cup K(b) = b \).

If \( A \subseteq L \), then \( \cup \{ \theta_a \mid a \in A \} = \theta_{\cup A} \). For, if \( C \in \Gamma \), then \( (\cup \{ \theta_a \mid a \in A \})(C) = \cup \{ \theta_a(C) \mid a \in A \} = \cup \{ f(C, a) \mid a \in A \} = \sum \{ f(C, a) \mid a \in A \} = f(C, \cup A) = \theta_{\cup A}(C) \). Also if \( A \subseteq L \), then \( \cap \{ \theta_a \mid a \in A \} = \theta_{\cap A} \). For, if \( C \in \Gamma \), then \( (\cap \{ \theta_a \mid a \in A \})(C) = \cap \{ \theta_a(C) \mid a \in A \} = \cap \{ f(C, a) \mid a \in A \} = f(C, \cap A) = \theta_{\cap A}(C) \).

It follows that \( L^* \) is a closed sublattice of \( D \), and that the mapping \( a \rightarrow \theta_a \) is a complete-isomorphism of \( L \) onto \( L^* \).

**References**


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