ON BOUNDS FOR THE GREATEST CHARACTERISTIC ROOT OF A MATRIX WITH POSITIVE ELEMENTS

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It is known that if $A = (a_{\kappa\lambda})$ is a square matrix of order n with positive elements, then the greatest characteristic root ω of A is positive, simple, and the characteristic vector X of this root may be chosen with all positive components $x_1, x_2, \cdots, x_n$ [3; 4].[2]

Let x_r be the greatest and x_p be the smallest component of X. It is sufficient to assume $x_r = 1$. A. Ostrowski [5] has just published the following bound for x_p.

If we set

$$R_\lambda = \sum_{\nu=1}^{n} a_{\lambda\nu} \qquad (\lambda = 1, 2, \cdots, n),$$

$$R = \max_{\lambda} R_\lambda,$$

$$r = \min_{\lambda} R_\lambda,$$

$$K = \min_{\mu \neq \nu} a_{\mu\nu},$$

then

$$x_p > \frac{K}{R - r + K} = \gamma$$

for $R \neq r$.

Using the above result we shall improve a theorem of Brauer [2, Theorem 39] for positive matrices. Following Brauer's notation, we let $A = (a_{\kappa\lambda})$ be a square matrix of order n with real elements. Assume that all the components of a characteristic vector belonging to the root ω are positive. Consider the quadratic form

$$\left(\sum_{\nu=1, \nu \neq \kappa}^{n} a_{\kappa\nu} y_\nu \right) \left(\sum_{\mu=1, \mu \neq \lambda}^{n} a_{\lambda\mu} y_\mu \right) \tag{1}$$

for given κ and λ, and combine the terms which contain the same product $y_\nu y_\mu$. Denote the sum of the positive coefficients in (1) by $T_{\kappa\lambda}$ and the sum of the negative coefficients by $t_{\kappa\lambda}$.

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Let

$$\max\left[(T_{\kappa\lambda} - \gamma^2 |t_{\kappa\lambda}|), (|t_{\kappa\lambda}| - \gamma^2 T_{\kappa\lambda})\right] = P_{\kappa\lambda}. \tag{2}$$

THEOREM. *Let $A=(a_{\kappa\lambda})$ be a square matrix of order n with positive elements and $f_1(y), f_2(y), \cdots, f_n(y)$ polynomials with real coefficients. Let $B=(b_{\kappa\lambda})$ be the matrix which has as the νth row the elements of the νth row of $f_\nu(A)$. If $P_{\kappa\lambda}^{(B)}$ are the numbers obtained from* (1) *and* (2) *for the matrix B, then the greatest characteristic root ω of A satisfies at least one of the inequalities*

$$|f_\kappa(\omega) - b_{\kappa\kappa}|\,|f_\lambda(\omega) - b_{\lambda\lambda}| \leqq P_{\kappa\lambda}^{(B)} \qquad (\kappa, \lambda = 1, 2, \cdots, n; \kappa \neq \lambda).$$

PROOF. The vector X is also a characteristic vector belonging to the characteristic root $f_\nu(\omega)$ of $f_\nu(A)$. For proof see Brauer [1, Lemma 2]. We have, therefore,

$$f_\nu(\omega)x_\nu = \sum_{\mu=1}^{n} b_{\nu\mu}x_\mu \qquad (\nu = 1, 2, \cdots, n)$$

or

$$(f_\nu(\omega) - b_{\nu\nu})x_\nu = \sum_{\mu=1,\mu\neq\nu}^{n} b_{\nu\mu}x_\mu \qquad (\nu = 1, 2, \cdots, n). \tag{3}$$

Assume now that x_s is such that

$$x_r \geqq x_s \geqq x_\nu \geqq x_p \qquad (\nu = 1, 2, \cdots, n; \nu \neq r, s, p).$$

Multiplying the rth and sth equations of the system (3), we have

$$[f_r(\omega) - b_{rr}][f_s(\omega) - b_{ss}]x_r x_s = \left(\sum_{\nu=1,\nu\neq r}^{n} b_{r\nu}x_\nu\right)\left(\sum_{\mu=1,\mu\neq s}^{n} b_{s\mu}x_\mu\right). \tag{4}$$

Let S_1 denote the sum of the positive terms and S_2 denote the sum of the negative terms in the right-hand side of (4), then

$$T_{rs}x_p^2 \leqq S_1 \leqq T_{rs}x_r x_s,$$

$$|t_{rs}|\,x_p^2 \leqq |S_2| \leqq |t_{rs}|\,x_r x_s.$$

Thus

$$|S_1 + S_2| \leqq \max\left[(T_{rs}x_r x_s - |t_{rs}|\,x_p^2), (|t_{rs}|\,x_r x_s - T_{rs}x_p^2)\right].$$

Therefore

$$|f_r(\omega) - b_{rr}|\,|f_s(\omega) - b_{ss}|\,x_r x_s \leqq \max\left[(T_{rs}x_r x_s - |t_{rs}|\,x_p^2), (|t_{rs}|\,x_r x_s - T_{rs}x_p^2)\right].$$

Since

$$x_p^2 > \gamma^2 \geqq x_r x_s \gamma^2,$$

we have

$$|f_r(\omega) - b_{rr}| \, |f_s(\omega) - b_{ss}| \, x_r x_s \leqq x_r x_s \max \left[(T_{rs} - \gamma^2 |t_{rs}|), (|t_{rs}| - \gamma^2 T_{rs}) \right].$$

Hence

$$|f_r(\omega) - b_{rr}| \, |f_s(\omega) - b_{ss}| \leqq P_{rs}^{(B)}.$$

It is evident that a similar theorem holds for the columns.

Bibliography

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