A SUFFICIENT CONDITION FOR A REGULAR MATRIX TO SUM A BOUNDED DIVERGENT SEQUENCE

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If a matrix A transforms a sequence $\{z_n\}$ into the sequence $\{\sigma_n\}$, i.e., if $\sigma_n = \sum_{k=1}^{\infty} a_{n,k}z_k$, and if $\sigma_n \to z$ as $n \to \infty$ whenever $z_n \to z$, A is said to be regular. The well known necessary and sufficient conditions for A to be regular are[1]

(a) $\sum_{k=1}^{\infty} |a_{n,k}| < M$ for every positive integer $n > n_0$,

(b) $\lim_{n\to\infty} a_{n,k} = 0$ for every fixed k,

(c) $\sum_{k=1}^{\infty} a_{n,k} \equiv A_n \to 1$ as $n \to \infty$.

It is known[2] that if a regular matrix sums a bounded divergent sequence, then it also sums some unbounded sequence. The converse is, however, false.[3] It is consequently of interest to find sufficient conditions for a regular matrix to sum a *bounded* divergent sequence. Many authors have considered summability of bounded sequences.[4] R. P. Agnew has given a simple sufficient condition that a regular matrix shall sum a bounded divergent sequence. He has proved[5] that if A is a regular matrix such that $\lim_{n,k\to\infty} a_{n,k} = 0$, then some divergent sequences of 0's and 1's are summable-A. There are, however, very many simple regular matrices which do not satisfy this condition, but which are known to sum a bounded divergent sequence. For example, the matrix A obtained by replacing every third row of the Cesàro matrix $(C, 1)$ by the corresponding row of the unit matrix, given by

$$a_{3n-2,k} = \frac{1}{3n-2} \quad (k \leqq 3n-2), \qquad a_{3n-1,k} = \frac{1}{3n-1} \quad (k \leqq 3n-1),$$

$$a_{3n,k} = \delta_{3n,k}, \qquad a_{n,k} = 0 \quad (k > n) \qquad\qquad (n, k = 1, 2, \cdots),$$

sums the sequence $\{0, 2, 1, 0, 2, 1, 0, \cdots\}$ to the limit 1. This matrix does, however, satisfy the conditions which will be given in Theorem II.

I first show that I need consider only normal matrices, i.e., lower-

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[1] See R. G. Cooke [1, pp. 64–65].

[2] Stated without proof by S. Mazur and W. Orlicz [2]; a proof is given by V. M. Darevsky [3]. See also J. D. Hill [4]; A. Wilansky [5]; K. Zeller [6].

[3] See R. G. Cooke [1, p. 178, Examples 7, no. 10].

[4] See, e.g., G. G. Lorentz [7; 8]; R. P. Agnew [9]; A. Wilansky [10; 11].

[5] R. P. Agnew [12, pp. 128–132]; this is a special case of G. G. Lorentz [7, p. 181, Theorem 8 and footnote].

semi-matrices with no zero element in the leading diagonal. A normal matrix has a unique right-hand reciprocal which is also normal, and which is also a left-hand reciprocal.[6] If a matrix B is such that $\|B\| = \sup_n \sum_k |b_{n,k}| = \infty$, then, by a method now classical, we can construct a null sequence whose B-transform is unbounded. It is not, in general, possible to construct a null sequence whose B-transform is bounded and divergent. This can be done, however, if B is normal and its columns form null sequences. This is the main result of this paper and its interest lies in its sufficiency that $B^{-1} = A$ shall sum a bounded divergent sequence.

The following theorem is due to A. Brudno.[7] Brudno's proof, however, is somewhat complicated, and I give here a simpler proof.

THEOREM I. *If A is a general (square) regular matrix, there exists a normal regular matrix A^*, such that A and A^* are mutually consistent*[8] *for bounded sequences.*

PROOF. Let $\{\epsilon_n\}$ be any null sequence with $\epsilon_n > 0$ for each n. Since A is regular, by (a) we can choose a monotonic increasing sequence of positive integers $\{p_n\}$ $(n = 1, 2, \cdots)$ such that

$$\sum_{k=p_n+1}^{\infty} |a_{n,k}| < \epsilon_n \qquad \text{for every } n.$$

Let the matrix A^* be given by

$$a^*_{n,k} = a_{1,k} \qquad (1 \leqq k < n < p_1),$$

$$a^*_{n,n} = \begin{cases} a_{1,n} & (a_{1,n} \neq 0) \\ 1/n & (a_{1,n} = 0) \end{cases} \qquad (n < p_1),$$

$$a^*_{n,k} = a_{l,k} \qquad (p_l \leqq n < p_{l+1},\ l \geqq 1,\ 1 \leqq k < n),$$

$$a^*_{n,n} = \begin{cases} a_{l,n} & (a_{l,n} \neq 0) \\ 1/n & (a_{l,n} = 0) \end{cases} \qquad (p_l \leqq n < p_{l+1},\ l \geqq 1),$$

$$a^*_{n,k} = 0 \qquad (k > n).$$

Let $\sigma_n = A(z_n) = \sum_{k=1}^{\infty} a_{n,k} z_k$, $\rho_n = A^*(z_n) = \sum_{k=1}^{n} a^*_{n,k} z_k$. If $p_l \leqq n < p_{l+1}$, $\sigma_l - \rho_n = \sum_{k=n+1}^{\infty} a_{l,k} z_k + (a_{l,n} - a^*_{n,n}) z_n$. Hence, if $\{z_n\}$ is a bounded sequence for which $|z_n| \leqq M$ for every n,

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[6] R. G. Cooke [1, pp. 19, 22].

[7] A. Brudno [13].

[8] I.e., A^* sums, to the same limit, every bounded sequence which is summable-A and vice versa.

$$\begin{aligned} | \sigma_l - \rho_n | &\leqq M \sum_{k=p_l+1}^{\infty} | a_{l,k} | + \frac{M}{n} \\ &< M\left(\epsilon_l + \frac{1}{n}\right) \to 0 \qquad \text{as } l \to \infty, \end{aligned}$$

since n and l tend to ∞ together.

Thus $A(z_n)$ and $A^*(z_n)$ either both converge to the same limit, or neither converges, and A^* is normal.

I now prove the main theorems.

THEOREM II. *In order that the regular normal matrix A shall sum a bounded divergent sequence it is sufficient that its unique two-sided reciprocal B shall not be regular, and that all the columns of B shall form bounded sequences.*

THEOREM III. *In order that the regular normal matrix A shall sum a bounded divergent sequence it is sufficient that*

(a) *its unique reciprocal B shall not be regular, and*

(b) *there exists a normal matrix Q with $\|Q\| < \infty$, whose columns are all null sequences, such that the matrix $C = BQ$ has bounded columns and $\|C\| = \infty$.*

PROOF OF THEOREM III. If $A(z_n) = \sigma_n$, then

$$B(\sigma_n) = B[A(z_n)] = (BA)(z_n) = (z_n),$$

the alteration in the order of summation being justified, since only finite sums are involved.

If B is regular, $\{z_n\}$ converges whenever $\{\sigma_n\}$ converges, so that A sums only convergent sequences. If B is not regular, there exists a convergent sequence $\{\sigma_n\}$ such that $\{z_n\}$ is divergent. Thus, in order that A shall be stronger than convergence it is necessary and sufficient that B shall not be regular.

Since B and Q are normal, $C = BQ$ is also normal, and hence

$$AC = A(BQ) = (AB)Q = Q,$$

so that, assuming condition (b), A transforms each column of C into a null sequence. Since A is regular, it follows that each column of C is either a divergent or a null sequence. If at least one column of C is divergent, the result is proved. There remains to be considered only the case in which all the columns of C form null sequences. Thus $c_{n,k} \to 0$ as $n \to \infty$ for every fixed k, and if $M_n = \sum_{k=1}^{n} |c_{n,k}|$, the sequence $\{M_n\}$ is unbounded, by hypothesis, and therefore has a subsequence which tends to infinity.

If $Z=re^{i\theta}$, let sgn $Z=e^{-i\theta}(Z\neq 0)$, sgn $0=0$.

Choose a positive integer n_1 such that $M_{n_1}>M_n$ for all $n<n_1$. Put

$$x_k = \frac{\operatorname{sgn}(c_{n_1,k})}{M_{n_1}} \qquad (k \leqq n_1).$$

If $C(x_n)=y_n$,

$$y_{n_1} = \sum_{k=1}^{n_1} c_{n_1,k}x_k = \frac{1}{M_{n_1}}\sum_{k=1}^{n_1}\left| c_{n_1,k}\right| = 1.$$

Let $\epsilon>0$ be fixed and arbitrarily small. We can choose $n_2>n_1$ such that

$$\sum_{k=1}^{n_1}\left| c_{n,k}\right| < \frac{1}{2}\epsilon \qquad \text{for every } n \geqq n_2$$

and

$$M_{n_2} > M_n \qquad \text{for every } n < n_2.$$

Put

$$x_k = -\frac{\operatorname{sgn}(c_{n_2,k})}{M_{n_2}} \qquad (n_1 < k \leqq n_2).$$

Then

$$\begin{aligned} y_{n_2} &= \sum_{k=1}^{n_1} c_{n_2,k}x_k - \sum_{k=n_1+1}^{n_2} \frac{c_{n_2,k}\operatorname{sgn}(c_{n_2,k})}{M_{n_2}} \\ &= \frac{1}{M_{n_1}}\sum_{k=1}^{n_1} c_{n_2,k}\operatorname{sgn}(c_{n_1,k}) - \frac{1}{M_n}\sum_{k=n_1+1}^{n_2}\left| c_{n_2,k}\right|. \end{aligned}$$

We now choose $n_3>n_2$ such that

$$\sum_{k=n_1+1}^{n_2}\left| c_{n,k}\right| < \frac{1}{2^2}\epsilon \qquad \text{for every } n \geqq n_3,$$

and

$$M_{n_3} > M_n \qquad \text{for every } n < n_3.$$

Put

$$x_k = \frac{\operatorname{sgn}(c_{n_3,k})}{M_{n_3}} \qquad (n_2 < k \leqq n_3).$$

Then

$$y_{n_2} = \frac{1}{M_{n_1}} \sum_{k=1}^{n_1} c_{n_2,k} \operatorname{sgn}(c_{n_1,k}) - \frac{1}{M_{n_2}} \sum_{k=n_1+1}^{n_2} c_{n_2,k} \operatorname{sgn}(c_{n_2,k}) + \frac{1}{M_{n_2}} \sum_{k=n_2+1}^{n_2} |c_{n_2,k}|.$$

Continue in this way; thus

$$x_k = (-1)^{p-1} \frac{\operatorname{sgn}(c_{n_p,k})}{M_{n_p}} \qquad (n_{p-1} < k \leqq n_p).$$

For any integer p,

$$\begin{aligned} 1 - \frac{1}{M_{n_p}} \sum_{k=n_{p-1}+1}^{n_p} |c_{n_p,k}| &= \frac{1}{M_{n_p}} \left\{ \sum_{k=1}^{n_1} |c_{n_p,k}| + \sum_{k=n_1+1}^{n_2} |c_{n_p,k}| + \cdots + \sum_{k=n_{p-2}+1}^{n_{p-1}} |c_{n_p,k}| \right\} \\ &< \frac{1}{M_{n_p}} \left\{ \frac{1}{2}\epsilon + \frac{1}{2^2}\epsilon + \cdots + \frac{1}{2^{p-1}}\epsilon \right\} \\ &< \frac{\epsilon}{M_{n_p}} \to 0 \qquad \text{as } p \to \infty, \end{aligned}$$

and is arbitrarily small for $p = 1, 2, 3, \cdots$. If p is odd,

$$\begin{aligned} &\left| y_{n_p} - \frac{1}{M_{n_p}} \sum_{k=n_{p-1}+1}^{n_p} |c_{n_p,k}| \right| \\ &< \frac{1}{M_{n_1}} \sum_{k=1}^{n_1} |c_{n_p,k}| + \frac{1}{M_{n_2}} \sum_{k=n_1+1}^{n_2} |c_{n_p,k}| + \cdots + \frac{1}{M_{n_{p-1}}} \sum_{k=n_{p-2}+1}^{n_{p-1}} |c_{n_p,k}| \\ &< \frac{1}{M_{n_1}} \cdot \frac{1}{2}\epsilon + \frac{1}{M_{n_2}} \cdot \frac{1}{2^2}\epsilon + \cdots + \frac{1}{M_{n_{p-1}}} \cdot \frac{1}{2^{p-1}}\epsilon \\ &< \frac{1}{M_{n_1}} \epsilon \left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{p-1}} \right) \\ &< \frac{\epsilon}{M_{n_1}}, \end{aligned}$$

which is arbitrarily small. The last two inequalities together show that

$y_{n_p} - 1$ can be made arbitrarily small when p is odd.

Similarly $y_{n_p} + 1$ can be made arbitrarily small when p is even. Thus the sequence $\{y_n\}$ is divergent. Moreover, if $n_q < n \leqq n_{q+1}$,

$$
\begin{aligned}
|y_n| &\leqq \frac{1}{M_{n_1}}\sum_{k=1}^{n_1}|c_{n,k}| + \frac{1}{M_{n_2}}\sum_{k=n_1+1}^{n_2}|c_{n,k}| + \cdots \\
&\quad + \frac{1}{M_{n_q}}\sum_{k=n_{q-1}+1}^{n_q}|c_{n,k}| + \frac{1}{M_{n_{q+1}}}\sum_{k=n_q+1}^{n}|c_{n,k}| \\
&< \frac{1}{M_{n_1}}\left\{\frac{1}{2}\epsilon + \frac{1}{2^2}\epsilon + \cdots + \frac{1}{2^q}\epsilon\right\} + \frac{M_n}{M_{n_{q+1}}} \\
&< \frac{\epsilon}{M_{n_1}} + 1, \quad \text{since} \quad M_n < M_{n_{q+1}}.
\end{aligned}
$$

Thus $\{y_n\}$ is a bounded divergent sequence, and $y_n = C(x_n)$, where $\{x_n\}$ is a null sequence.

Hence $B[Q(x_n)] = (BQ)(x_n) = C(x_n) = y_n$. Let $Q(x_n) = \xi_n$. Now since $\|Q\| < \infty$ and $q_{n,k} \to 0$ as $n \to \infty$ for every fixed k, it follows[9] that Q transforms every null sequence into a null sequence. Thus $\{\xi_n\}$ is a null sequence and $B(\xi_n) = y_n$. Hence $A(y_n) = \xi_n$, and A sums the bounded divergent sequence $\{y_n\}$ to the limit zero.

The theorem is now proved.

For $Q = I$, Theorem II follows. For, in this case, $M_n = \sum_{k=1}^{n} |b_{n,k}|$. It is obvious that the sequence $\{M_n\}$ is unbounded; for if $M_n < M$ for every n, B would transform every convergent sequence into a bounded sequence. This would imply that all the divergent sequences which are summable-A are bounded. This is impossible, as already mentioned.

COROLLARY. *The theorem still holds if all but a finite number of the columns of C form bounded sequences.*

If all but the first N columns are bounded, we put $x_k = 0$ $(k \leqq N)$. Define $\{M_n\}$ by the equation $M_n = \sum_{k=N+1}^{n} |c_{n,k}|$ $(n > N)$, and with slight modifications the proof proceeds as before.

EXAMPLES. The matrix A, already quoted, obtained by modifying the $(C, 1)$ matrix, has reciprocal B given by

$$
\begin{aligned}
&b_{3n,3n} = 1, \qquad b_{3n-1,3n-1} = 3n - 1, \qquad b_{3n-1,3n-2} = -(3n - 2), \\
&b_{3n-2,3n-2} = 3n - 2, \qquad b_{3n-2,3n-3} = -1, \\
&b_{3n-2,3n-4} = -(3n - 4), \qquad b_{n,k} = 0 \text{ otherwise.}
\end{aligned}
$$

B is not regular, and every column of B tends to zero. The conditions of Theorem II are satisfied.

[9] See, e.g., R. G. Cooke [1, p. 64].

P. Vermes has suggested the following example of a matrix which satisfies the conditions of Theorem III.

Let U be the matrix for which $u_{n+1,n}=1$, $u_{n,k}=0$ otherwise. Take $A=2^{-p}(I+U)^p$, p being a positive integer $\geqq 2$; then A is regular, and sums the sequence $\{1, 0, 1, 0, 1, 0, \cdots\}$ to $1/2$. $B=2^p(I+U)^{-p}$ is not regular and its columns are not bounded. Take $Q=(I+U)^{p-1}$; then $\|Q\|=2^{p-1}$ and Q has zero column limits. Thus $C=BQ=2^p(I+U)^{-1}$, which has bounded columns, and $\|C\|=\infty$.

I am unable to prove that the conditions of Theorem III are also necessary.

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