A SUFFICIENT CONDITION FOR A REGULAR MATRIX TO SUM A BOUNDED DIVERGENT SEQUENCE

A. MARY TROPPER

If a matrix \( A \) transforms a sequence \( \{ z_n \} \) into the sequence \( \{ s_n \} \), i.e., if \( s_n = \sum_{k=1}^{n} a_{n,k} z_k \), and if \( s_n \to z \) as \( n \to \infty \) whenever \( z_n \to z \), \( A \) is said to be regular. The well known necessary and sufficient conditions for \( A \) to be regular are:

(a) \( \sum_{n=1}^{\infty} |a_{n,k}| < M \) for every positive integer \( n > n_0 \),
(b) \( \lim_{n \to \infty} a_{n,k} = 0 \) for every fixed \( k \),
(c) \( \sum_{k=1}^{\infty} a_{n,k} \equiv A \to 1 \) as \( n \to \infty \).

It is known that if a regular matrix sums a bounded divergent sequence, then it also sums some unbounded sequence. The converse is, however, false. It is consequently of interest to find sufficient conditions for a regular matrix to sum a bounded divergent sequence. Many authors have considered summability of bounded sequences. R. P. Agnew has given a simple sufficient condition that a regular matrix shall sum a bounded divergent sequence. He has proved that if \( A \) is a regular matrix such that \( \lim_{n \to \infty} a_{n,k} = 0 \), then some divergent sequences of 0's and 1's are summable-\( A \). There are, however, very many simple regular matrices which do not satisfy this condition, but which are known to sum a bounded divergent sequence. For example, the matrix \( A \) obtained by replacing every third row of the Cesàro matrix \( (C, 1) \) by the corresponding row of the unit matrix, given by

\[
\begin{align*}
    a_{3n-2,k} &= \frac{1}{3n-2} \quad (k \leq 3n-2), \\
    a_{3n-1,k} &= \frac{1}{3n-1} \quad (k \leq 3n-1), \\
    a_{3n,k} &= \delta_{3n,k}, \\
    a_{n,k} &= 0 \quad (k > n)
\end{align*}
\]

sums the sequence \( \{ 0, 2, 1, 0, 2, 1, 0, \ldots \} \) to the limit 1. This matrix does, however, satisfy the conditions which will be given in Theorem II.

I first show that I need consider only normal matrices, i.e., lower-

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1 See R. G. Cooke [1, pp. 64-65].
2 Stated without proof by S. Mazur and W. Orlicz [2]; a proof is given by V. M. Darevsky [3]. See also J. D. Hill [4]; A. Wilansky [5]; K. Zeller [6].
3 See R. G. Cooke [1, p. 178, Examples 7, no. 10].
4 See, e.g., G. G. Lorentz [7; 8]; R. P. Agnew [9]; A. Wilansky [10; 11].
5 R. P. Agnew [12, pp. 128-132]; this is a special case of G. G. Lorentz [7, p. 181, Theorem 8 and footnote].

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semi-matrices with no zero element in the leading diagonal. A normal matrix has a unique right-hand reciprocal which is also normal, and which is also a left-hand reciprocal.\footnote{R. G. Cooke [1, pp. 19, 22].} If a matrix $B$ is such that $\|B\| = \sup_n \sum_k |b_{n,k}| = \infty$, then, by a method now classical, we can construct a null sequence whose $B$-transform is unbounded. It is not, in general, possible to construct a null sequence whose $B$-transform is bounded and divergent. This can be done, however, if $B$ is normal and its columns form null sequences. This is the main result of this paper and its interest lies in its sufficiency that $B^{-1}=A$ shall sum a bounded divergent sequence.

The following theorem is due to A. Brudno.\footnote{A. Brudno [13].} Brudno's proof, however, is somewhat complicated, and I give here a simpler proof.

**Theorem I.** If $A$ is a general (square) regular matrix, there exists a normal regular matrix $A^*$, such that $A$ and $A^*$ are mutually consistent\footnote{I.e., $A^*$ sums, to the same limit, every bounded sequence which is summable-$A$ and vice versa.} for bounded sequences.

**Proof.** Let $\{e_n\}$ be any null sequence with $e_n > 0$ for each $n$. Since $A$ is regular, by (a) we can choose a monotonic increasing sequence of positive integers $\{p_n\}$ ($n = 1, 2, \cdots$) such that

$$\sum_{k=p_n+1}^{\infty} |a_{n,k}| < e_n$$

for every $n$.

Let the matrix $A^*$ be given by

$$a_{n,k}^* = a_{1,k} \quad (1 \leq k < n < p_1),$$

$$a_{n,n}^* = \begin{cases} a_{1,n} & (a_{1,n} \neq 0) \\ 1/n & (a_{1,n} = 0) \end{cases} \quad (n < p_1),$$

$$a_{n,k}^* = a_{1,k} \quad (p_1 \leq n < p_{l+1}, l \geq 1, 1 \leq k < n),$$

$$a_{n,n}^* = \begin{cases} a_{1,n} & (a_{1,n} \neq 0) \\ 1/n & (a_{1,n} = 0) \end{cases} \quad (p_1 \leq n < p_{k+1}, l \geq 1),$$

$$a_{n,k}^* = 0 \quad (k > n).$$

Let $\sigma_n = A(z_n) = \sum_{k=1}^{n} a_{n,k}z_k$, $\rho_n = A^*(z_n) = \sum_{k=1}^{n} a_{n,k}^*z_k$. If $p_1 \leq n < p_{l+1}$, $\sigma_l - \rho_n = \sum_{k=n+1}^{\infty} a_{1,k}z_k + (a_{1,n} - a_{n,n}^*)z_n$. Hence, if $\{z_n\}$ is a bounded sequence for which $|z_n| \leq M$ for every $n$,
\[ |\sigma_1 - \rho_n| \leq M \sum_{k=p+1}^{\infty} |a_{1,k}| + \frac{M}{n} \]
\[ < M \left( \epsilon_1 + \frac{1}{n} \right) \to 0 \quad \text{as } n \to \infty, \]

since \( n \) and \( l \) tend to \( \infty \) together.

Thus \( A(z_n) \) and \( A^*(z_n) \) either both converge to the same limit, or neither converges, and \( A^* \) is normal.

I now prove the main theorems.

**Theorem II.** In order that the regular normal matrix \( A \) shall sum a bounded divergent sequence it is sufficient that its unique two-sided reciprocal \( B \) shall not be regular, and that all the columns of \( B \) shall form bounded sequences.

**Theorem III.** In order that the regular normal matrix \( A \) shall sum a bounded divergent sequence it is sufficient that

(a) its unique reciprocal \( B \) shall not be regular, and

(b) there exists a normal matrix \( Q \) with \( ||Q|| < \infty \), whose columns are all null sequences, such that the matrix \( C = BQ \) has bounded columns and \( ||C|| = \infty \).

**Proof of Theorem III.** If \( A(z_n) = \sigma_n \), then

\[ B(\sigma_n) = B[A(z_n)] = (BA)(\sigma_n) = (\sigma_n), \]

the alteration in the order of summation being justified, since only finite sums are involved.

If \( B \) is regular, \( \{z_n\} \) converges whenever \( \{\sigma_n\} \) converges, so that \( A \) sums only convergent sequences. If \( B \) is not regular, there exists a convergent sequence \( \{\sigma_n\} \) such that \( \{z_n\} \) is divergent. Thus, in order that \( A \) shall be stronger than convergence it is necessary and sufficient that \( B \) shall not be regular.

Since \( B \) and \( Q \) are normal, \( C = BQ \) is also normal, and hence

\[ AC = A(BQ) = (AB)Q = Q, \]

so that, assuming condition (b), \( A \) transforms each column of \( C \) into a null sequence. Since \( A \) is regular, it follows that each column of \( C \) is either a divergent or a null sequence. If at least one column of \( C \) is divergent, the result is proved. There remains to be considered only the case in which all the columns of \( C \) form null sequences. Thus \( c_{n,k} \to 0 \) as \( n \to \infty \) for every fixed \( k \), and if \( M_n = \sum_{k=1}^{n} |c_{n,k}| \), the sequence \( \{M_n\} \) is unbounded, by hypothesis, and therefore has a subsequence which tends to infinity.
If $Z = r e^{i \theta}$, let $\text{sgn } Z = e^{-\theta} (Z \neq 0)$, $\text{sgn } 0 = 0$.
Choose a positive integer $n_1$ such that $M_{n_1} > M_n$ for all $n < n_1$. Put

$$x_k = \frac{\text{sgn } (c_{n_1,k})}{M_{n_1}} \quad (k \leq n_1).$$

If $C(x_n) = y_n$,

$$y_{n_1} = \sum_{k=1}^{n_1} c_{n_1,k} x_k = \frac{1}{M_{n_1}} \sum_{k=1}^{n_1} |c_{n_1,k}| = 1.$$

Let $\epsilon > 0$ be fixed and arbitrarily small. We can choose $n_2 > n_1$ such that

$$\sum_{k=1}^{n_1} |c_{n_1,k}| < \frac{1}{2} \epsilon \quad \text{for every } n \geq n_3$$

and

$$M_{n_2} > M_n \quad \text{for every } n < n_2.$$

Put

$$x_k = -\frac{\text{sgn } (c_{n_2,k})}{M_{n_2}} \quad (n_1 < k \leq n_2).$$

Then

$$y_{n_2} = \sum_{k=1}^{n_1} c_{n_2,k} x_k - \sum_{k=n_2+1}^{n_3} c_{n_2,k} \text{sgn } (c_{n_2,k}) = \frac{1}{M_{n_1}} \sum_{k=1}^{n_1} |c_{n_2,k}| - \frac{1}{M_n} \sum_{k=n_2+1}^{n_3} |c_{n_2,k}|.$$

We now choose $n_3 > n_2$ such that

$$\sum_{k=n_2+1}^{n_3} |c_{n_2,k}| < \frac{1}{2} \epsilon \quad \text{for every } n \geq n_3,$$

and

$$M_{n_3} > M_n \quad \text{for every } n < n_3.$$

Put

$$x_k = \frac{\text{sgn } (c_{n_3,k})}{M_{n_3}} \quad (n_2 < k \leq n_3).$$

Then
\[ y_n = - \frac{1}{M_n} \sum_{k=1}^{n} c_{nk, k} \text{sgn} (e_{n1, k}) - \frac{1}{M_n} \sum_{k=n+1}^{n} c_{nk, k} \text{sgn} (e_{n2, k}) \]
\[ + \frac{1}{M_n} \sum_{k=n+1}^{\infty} |c_{nk, k}|. \]

Continue in this way; thus

\[ x_k = (-1)^{p-1} \frac{\text{sgn} (c_{np, k})}{M_p} \quad (n_p < k \leq n_p). \]

For any integer \( p \),

\[ 1 - \frac{1}{M_n} \sum_{k=n_{p-1}+1}^{n_p} |c_{np, k}| = \frac{1}{M_n} \left\{ \sum_{k=1}^{n} |c_{np, k}| + \sum_{k=n+1}^{n} |c_{np, k}| + \cdots \right. \]
\[ + \frac{1}{M_{n_{p-1}}} \sum_{k=n_{p-1}+1}^{n_p} |c_{np, k}| \left\} \right. \]
\[ < \frac{1}{M_n} \left\{ \frac{1}{2} + \frac{1}{2^2} \epsilon + \cdots + \frac{1}{2^{p-1}} \epsilon \right\} \]
\[ < \frac{\epsilon}{M_n} \to 0 \quad \text{as} \ p \to \infty, \]

and is arbitrarily small for \( p = 1, 2, 3, \ldots \). If \( p \) is odd,

\[ y_{n_p} - \frac{1}{M_n} \sum_{k=n_{p-1}+1}^{n_p} |c_{np, k}| \]
\[ < \frac{1}{M_n} \sum_{k=1}^{n} |c_{np, k}| + \frac{1}{M_n} \sum_{k=n+1}^{n} |c_{np, k}| + \cdots + \frac{1}{M_{n_{p-1}}} \sum_{k=n_{p-1}+1}^{n_p} |c_{np, k}| \]
\[ < \frac{1}{M_n} \frac{1}{2} \epsilon + \frac{1}{M_n} \frac{1}{2^2} \epsilon + \cdots + \frac{1}{M_{n_{p-1}}} \frac{1}{2^{p-1}} \epsilon \]
\[ < \frac{1}{M_n} \epsilon \left( \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{p-1}} \right) \]
\[ < \frac{\epsilon}{M_n}, \]

which is arbitrarily small. The last two inequalities together show that

\[ y_{n_p} - 1 \text{ can be made arbitrarily small when } p \text{ is odd.} \]

Similarly \( y_{n_p} + 1 \) can be made arbitrarily small when \( p \) is even. Thus the sequence \( \{y_n\} \) is divergent. Moreover, if \( n \leq n_{p+1} \),
Thus \( \{y_n\} \) is a bounded divergent sequence, and \( y_n = C(x_n) \), where \( \{x_n\} \) is a null sequence.

Hence \( B[Q(x_n)] = (BQ)(x_n) = C(x_n) = y_n \). Let \( Q(x_n) = \xi_n \). Now since \( ||Q|| < \infty \) and \( \xi_n, k \to 0 \) as \( n \to \infty \) for every fixed \( k \), it follows that \( Q \) transforms every null sequence into a null sequence. Thus \( \{\xi_n\} \) is a null sequence and \( B(\xi_n) = y_n \). Hence \( A(y_n) = \xi_n \), and \( A \) sums the bounded divergent sequence \( \{y_n\} \) to the limit zero.

The theorem is now proved.

For \( Q = I \), Theorem II follows. For, in this case, \( M_n = \sum_{k=1}^{n} |b_{n,k}| \).

It is obvious that the sequence \( \{M_n\} \) is unbounded; for if \( M_n < M \) for every \( n \), \( B \) would transform every convergent sequence into a bounded sequence. This would imply that all the divergent sequences which are summable-\( A \) are bounded. This is impossible, as already mentioned.

**Corollary.** The theorem still holds if all but a finite number of the columns of \( C \) form bounded sequences.

If all but the first \( N \) columns are bounded, we put \( x_k = 0 \) (\( k \leq N \)). Define \( \{M_n\} \) by the equation \( M_n = \sum_{k=N+1}^{n} |c_{n,k}| \) (\( n > N \)), and with slight modifications the proof proceeds as before.

**Examples.** The matrix \( A \), already quoted, obtained by modifying the \((C, 1)\) matrix, has reciprocal \( B \) given by

\[
\begin{align*}
b_{2n,3n} &= 1, & b_{3n-1,3n-1} &= 3n - 1, & b_{2n-1,3n-2} &= -(3n - 2), \\
b_{2n-2,3n-2} &= 3n - 2, & b_{2n-2,3n-3} &= -1, \\
b_{2n-2,3n-4} &= -(3n - 4), & b_{n,k} &= 0 \text{ otherwise.}
\end{align*}
\]

\( B \) is not regular, and every column of \( B \) tends to zero. The conditions of Theorem II are satisfied.

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\* See, e.g., R. G. Cooke [1, p. 64].
P. Vermes has suggested the following example of a matrix which satisfies the conditions of Theorem III.

Let $U$ be the matrix for which $u_{n+1,n} = 1$, $u_{n,k} = 0$ otherwise. Take $A = 2^{-p}(I + U)^p$, $p$ being a positive integer $\geq 2$; then $A$ is regular, and sums the sequence $\{1, 0, 1, 0, 1, 0, \cdots \}$ to $1/2$. $B = 2^{p}(I + U)^{-p}$ is not regular and its columns are not bounded. Take $Q = (I + U)^{p-1}$; then $\|Q\| = 2^{p-1}$ and $Q$ has zero column limits. Thus $C = BQ = 2^{p}(I + U)^{-1}$, which has bounded columns, and $\|C\| = \infty$.

I am unable to prove that the conditions of Theorem III are also necessary.

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References


Queen Mary College, University of London