LIE ALGEBRAS OF TYPE F

MARVIN L. TOMBER[1]

Chevalley and Schafer [4][2] have shown that the exceptional simple Lie algebra F_4 of dimension 52 over an arbitrary algebraically closed field Ω of characteristic 0 is the derivation algebra of the unique exceptional simple Jordan algebra of dimension 27 over Ω. In this paper we show that a Lie algebra $\mathfrak{L}$ over an arbitrary field Φ of characteristic 0 is of type F if and only if $\mathfrak{L}$ is isomorphic to the derivation algebra $\mathfrak{D}(\mathfrak{J})$ of an exceptional central simple Jordan algebra $\mathfrak{J}$ over Φ. The proof given for this theorem requires a characterization of the automorphisms of $\mathfrak{D}(\mathfrak{J})$ over Ω. We prove that every automorphism of $\mathfrak{D}(\mathfrak{J})$ has the form $D\rightarrow SDS^{-1}$ for a unique automorphism S of $\mathfrak{J}$. The classification of Lie algebras of type F over Φ is reduced to the problem of classifying exceptional central simple Jordan algebras over Φ, since it is shown that $\mathfrak{D}(\mathfrak{J}_1)\cong\mathfrak{D}(\mathfrak{J}_2)$ if and only if $\mathfrak{J}_1\cong\mathfrak{J}_2$. In the last section of this paper the three exceptional central simple Jordan algebras over a real closed field are exhibited and their derivation algebras are the real closed Lie algebras of type F.

1. **Exceptional central simple Jordan algebras.** Let Ω be an algebraically closed field of characteristic 0. The exceptional simple Jordan algebra $\mathfrak{J}$ over Ω is the nonassociative algebra of dimension 27 whose elements are 3×3 Hermitian matrices with elements in the unique Cayley algebra $\mathfrak{C}$ of dimension 8 over Ω. Thus the elements of $\mathfrak{J}$ have the form

$$x = \begin{pmatrix} \xi_1 & c_3 & \bar{c}_2 \\ \bar{c}_3 & \xi_2 & c_1 \\ c_2 & \bar{c}_1 & \xi_3 \end{pmatrix},$$

ξ_i in Ω and c_i, $\bar{c}_i$ in $\mathfrak{C}$ $(i=1, 2, 3)$ where $\bar{c}_i$ is the conjugate of c_i [8, p. 83]. Multiplication in $\mathfrak{J}$ is defined as $xy=(x \circ y+y \circ x)/2$ where $x \circ y$ is the ordinary matrix product. Let e_i be the matrix with $\xi_i=1$, all other entries 0, and $\mathfrak{X}_i$ be the set of matrices with all entries

Presented to the Society, April 26, 1952; received by the editors March 4, 1953.

[1] This paper is the essential portion of the author's doctoral dissertation, which was written at the University of Pennsylvania. The author wishes to thank Professor R. D. Schafer who suggested this problem. His valuable assistance and kind encouragement are greatly appreciated.

[2] Numbers in brackets refer to references at the end of the paper.

0 except c_i and $\bar{c}_i$ ($i=1, 2, 3$). In this section i, j, k will be a permutation of 1, 2, 3. Elements of $\mathfrak{T}_i$ will be denoted by t_i or t_i'. Clearly multiplication is commutative and is such that; $e_i^2=e_i$, $e_ie_j=0$, $e_it_i=0$, $e_it_j=t_j/2$, t_it_i' in $\Omega(e_j+e_k)$, and t_it_j in $\mathfrak{T}_k$.

The derivation algebra $\mathfrak{D}(\mathfrak{J})$ of $\mathfrak{J}$ is the Lie algebra of linear transformations D on $\mathfrak{J}$ satisfying

$$D(xy) = (Dx)y + x(Dy).$$

Chevalley and Schafer have proved [4] that $\mathfrak{D}(\mathfrak{J})$ is the exceptional simple Lie algebra F_4 of dimension 52 and rank 4 over Ω. Let $\mathfrak{D}_0$ be the subalgebra of derivations which map e_1, e_2, and e_3 into 0, and $\mathfrak{D}_i$ the subalgebra which maps e_i into 0. Since $D1=0$, $D_ie_j=-D_ie_k$ for D_i in $\mathfrak{D}_i$. In [4] it is shown that for any D_0 in $\mathfrak{D}_0$, D_0t_i is in $\mathfrak{T}_i$, and that for any t_i in $\mathfrak{T}_i$ there is a D_i in $\mathfrak{D}_i$ such that $D_ie_j=t_i$.

Associated with the algebra $\mathfrak{J}$ is a symmetric nondegenerate bilinear form Sp xy where Sp $x=\xi_1+\xi_2+\xi_3$. This bilinear form is left invariant by any derivation, i.e. Sp $(Dx)y+$Sp $x(Dy)=0$. The set of elements x such that Sp $x=0$ form a subspace $\mathfrak{J}_0$ of $\mathfrak{J}$ of dimension 26. $\mathfrak{J}_0$ is an irreducible representation space of $\mathfrak{D}=\mathfrak{D}(\mathfrak{J})$. We denote the restriction of D to $\mathfrak{J}_0$ also by D. If R is any linear transformation on $\mathfrak{J}_0$ which commutes with all D in $\mathfrak{D}$, then $R=\sigma I$, for by Schur's Lemma the set of linear transformations which commute with $\mathfrak{D}$ form a division algebra containing I (the identity linear transformation on $\mathfrak{J}_0$) and since Ω is algebraically closed this set is ΩI.

Since Sp xy is a nondegenerate bilinear form we may define the adjoints A^* of any linear transformation A on $\mathfrak{J}$ by

$$\text{Sp } (A^*x)y = \text{Sp } x(Ay), \qquad \text{for all } x, y \in \mathfrak{J}.$$

The restriction of Sp xy to $\mathfrak{J}_0\times\mathfrak{J}_0$ is also a nondegenerate bilinear form. For any linear transformation B on $\mathfrak{J}_0$ we similarly define the adjoint B^*, on $\mathfrak{J}_0$, of B. Since Sp $(Dx_0)y_0=-$Sp $x_0(Dy_0)$, for all x_0, y_0 in $\mathfrak{J}_0$, $D^*=-D$ for any D in $\mathfrak{D}$. The mapping $A\to A^*$ is an involutorial anti-automorphism in the algebra of linear transformations on $\mathfrak{J}_0$.

Let Φ be a field of characteristic 0. The exceptional simple Jordan algebras over Φ are those simple Jordan algebras which are of degree 3 and dimension 27 over their centers. Exceptional central simple Jordan algebras over Φ have been characterized by Schafer [8]. They are the algebras $\mathfrak{J}=\mathfrak{H}(\mathfrak{C}, p)$ of 3×3 matrices x with elements in a Cayley algebra $\mathfrak{C}$ over Φ satisfying $x=p\bar{x}'p^{-1}$, p a nonsingular diagonal matrix in Φ_3 and $\bar{x}'$ the conjugate transpose of x. Multiplication in $\mathfrak{J}$ is defined by $xy=(x \circ y+y \circ x)/2$, where $x \circ y$ is the ordinary matrix multiplication. For x, y, z in $\mathfrak{J}$ the associator $A(x, y, z)$ is defined as $A(x, y, z)=(xy)z-x(yz)$. The subspace $\mathfrak{P}$ spanned by all associators is called the associator subspace of $\mathfrak{J}$, and it is known [9] that $\mathfrak{J}$ is the direct sum $\mathfrak{J}=\Phi 1+\mathfrak{P}$. Since Sp $(xy)z=$ Sp $x(yz)$ for x, y, z in $\mathfrak{J}$, we have Sp $x=0$ for all x in $\mathfrak{P}$. From the direct sum decomposition of $\mathfrak{J}$, $\mathfrak{P}$ has dimension 26 and hence $\mathfrak{P}$ is the set $\mathfrak{J}_0$ of all x in $\mathfrak{J}$ for which Sp $x=0$.

From this characterization of $\mathfrak{J}_0$, it is easy to see that $S\mathfrak{J}_0=\mathfrak{J}_0$ for any automorphism S of $\mathfrak{J}$, since $\mathfrak{J}_0$ is spanned by the elements $A(Sx, Sy, Sz)$. Also $(J_0)_\Sigma=(J_\Sigma)_0$ for any extension Σ of Φ.

It is known that every derivation D of $\mathfrak{J}$ is inner; that is, D has the form $y\rightarrow Dy=\Sigma A(x, y, z)$ [7, Theorem 2]. Hence $\mathfrak{D}\mathfrak{J}=\mathfrak{J}_0$. Moreover $\mathfrak{J}_0$ is an irreducible representation space for $\mathfrak{D}$. For if $\mathfrak{M}$ is invariant with respect to $\mathfrak{D}$, then $\mathfrak{M}_\Omega$ is invariant with respect to $\mathfrak{D}_\Omega$. But it is known [4, p. 141] that $(\mathfrak{J}_0)_\Omega$ is an irreducible representation space for $\mathfrak{D}_\Omega$, Ω the algebraic closure of Φ.

2. **Similarity of representations of F_4.** Let Ω be an arbitrary algebraically closed field of characteristic 0. F_4 is the exceptional simple Lie algebra of dimension 52 over Ω. In [2] it is shown that if $\Lambda=m_1\lambda_1+m_2\lambda_2+m_3\lambda_3+m_4\lambda_4$ is a weight of a representation P of F_4, then $2m_i$, $m_i\pm m_j$, and $m_1\pm m_2\pm m_3\pm m_4$ are integers and the linear forms:

$$\Lambda - \lambda_i,\ \Lambda - 2\lambda_i,\ \cdots,\ \Lambda - 2m_i\lambda_i, \tag{1}$$

$$\Lambda - (\lambda_i \pm \lambda_j),\ \cdots,\ \Lambda - (m_i \pm m_j)(\lambda_i \pm \lambda_j), \tag{2}$$

$$\Lambda - (\lambda_1 \pm \lambda_2 \pm \lambda_3 \pm \lambda_4)/2,\ \cdots,\ \Lambda - ((m_1 \pm m_2 \pm m_3 \pm m_4)/2)(\lambda_1 \pm \lambda_2 \pm \lambda_3 \pm \lambda_4) \tag{3}$$

$(i, j=1, \cdots, 4)$ are also weights of P. From (1) it can be seen that if Λ is a weight then so is $-\Lambda$, from (2) that if Λ is a weight then so is Λ' where Λ' is obtained from Λ by a permutation of the m_i, and (1), (2), and (3) give that for a highest weight:

$$m_1 \geqq m_2 \geqq m_3 \geqq m_4 \geqq 0, \tag{4}$$

$$m_1 \geqq m_2 + m_3 + m_4. \tag{5}$$

It is shown in [4] that there is an irreducible representation of F_4 of degree 26.

LEMMA. *Any two irreducible representations of degree* 26 *of the Lie algebra F_4 are similar.*

For the proof it is sufficient to show [2] that any two irreducible representations of F_4 of degree 26 have the same highest weight. Cartan has also shown in [2] that the number of weights of a representation does not exceed the degree of the representation. From (4) and (5) it may be seen that if Λ is the highest weight then Λ is in one of the following:

(i) At least three distinct $m_i > 0$,

(ii) $m_1 > m_2 = m_3 = m_4 > 0$,

(iii) $m_1 = m_2 > m_3 = m_4 = 0$,

(iv) $m_1 > m_2 = m_3 = m_4 = 0$,

(v) $m_1 = m_2 = m_3 = m_4 = 0$.

Note that (v) would imply that the representation is zero and therefore reducible, which is a contradiction. Cases (i), (ii), and (iii) may be eliminated, for by using the properties of weights of F_4, as given above, it may be seen that in these cases there would be more than 26 distinct weights. In (iv), $\Lambda = m\lambda_1$, m a positive integer. If $m \geqq 2$ then there are again more than 26 distinct weights, thus the only possible highest weight is $\Lambda = \lambda_1$.

3. **Automorphisms of F_4.** This section is devoted to the proof of the following theorem which characterizes the automorphisms of the exceptional Lie algebra F_4 of dimension 52 over Ω.

THEOREM 1. *If $D \rightarrow D^{\tilde{S}}$ is an automorphism of $\mathfrak{D}(\mathfrak{J})$, $\mathfrak{J}$ the exceptional simple Jordan algebra over an algebraically closed field Ω of characteristic 0, then there is a unique automorphism S of $\mathfrak{J}$ such that $D^{\tilde{S}} = SDS^{-1}$.*

The automorphism $\tilde{S}$ defines a second irreducible representation of the Lie algebra $\mathfrak{D} = \mathfrak{D}(\mathfrak{J})$ acting on $\mathfrak{J}_0$. By the lemma of §2 there is a nonsingular linear transformation S_1 on $\mathfrak{J}_0$ such that for D and $D^{\tilde{S}}$ on $\mathfrak{J}_0$, $D^{\tilde{S}} = S_1 D S_1^{-1}$. Then $D^{\tilde{S}} = (S_1^*)^{-1} D S_1^*$ or $D = S_1^* S_1 D (S_1^* S_1)^{-1}$. Thus $S_1^* S_1$ commutes with every D in $\mathfrak{D}$ and $S_1^* S_1 = \sigma I$, $\sigma \neq 0$ in Ω. Let $S_2 = \sigma^{-1/2} S_1$, where we reserve until later the choice of which square root of σ we use. Define a linear transformation S on $\mathfrak{J}$ as follows:

$$Sx = S(\gamma 1 + x_0) = \gamma 1 + S_2 x_0, \qquad \gamma \in \Omega,\ x_0 \in \mathfrak{J}_0.$$

S is a nonsingular linear transformation on $\mathfrak{J}$ and the mapping $D \rightarrow SDS^{-1}$ of $\mathfrak{D}$ is the same as $D \rightarrow D^{\tilde{S}}$. It is easily seen that $S^* x = S^*(\gamma 1 + x_0) = \gamma 1 + S_2^* x_0$ for all x in $\mathfrak{J}$. Hence $S^* S = I$, the identity on $\mathfrak{J}$, and

$$\text{Sp}\,(Sx)(Sy) = \text{Sp}\,(S^* Sx) y = \text{Sp}\, xy. \tag{6}$$

Furthermore since $S1=1$,

$$\text{Sp}\, Sx = \text{Sp}\,(Sx)(S1) = \text{Sp}\, x. \tag{7}$$

We shall show that S is an automorphism of $\mathfrak{J}$. Let i, j, k be a permutation of 1, 2, 3 for the remainder of this section. We begin by showing that the elements $f_i = Se_i$ are pairwise orthogonal idempotents of $\mathfrak{J}$. We write $u_i = St_i$. Since S is a nonsingular linear transformation on $\mathfrak{J}$, any element y of $\mathfrak{J}$ may be written uniquely as $y = \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 + u_1 + u_2 + u_2$. Let

$$f_i^2 = \alpha_{ii} f_i + \alpha_{ij} f_j + \alpha_{ik} f_k + u_i^{(i)} + u_j^{(i)} + u_k^{(i)}. \tag{8}$$

For any D_0 in $\mathfrak{D}_0$ we apply the derivation $D_0^{\tilde{S}}$ to (8) to obtain

$$\begin{aligned} 2f_i(D_0^{\tilde{S}} f_i) &= 2f_i(SD_0 e_i) = 0 \\ &= SD_0(\alpha_{ii} e_i + \alpha_{ij} e_j + \alpha_{ik} e_k + t_i^{(i)} + t_j^{(i)} + t_k^{(i)}) \\ &= SD_0(t_i^{(i)} + t_j^{(i)} + t_k^{(i)}). \end{aligned}$$

Since S is nonsingular:

$$D_0(t_i^{(i)} + t_j^{(i)} + t_k^{(i)}) = 0.$$

For all D_0 in $\mathfrak{D}_0$, $D_0 t_h$ is in $\mathfrak{T}_h$, $h=1, 2, 3$, hence $D_0 t_h^{(i)} = 0$ for any D_0 in $\mathfrak{D}_0$. Now $\mathfrak{D}_0$ on $\mathfrak{T}_h$ is the orthogonal Lie algebra $\mathfrak{o}(8, \Omega)$ [4]. The associative enveloping algebra of $\mathfrak{o}(8, \Omega)$ on a space of dimension 8 is the full matrix algebra Ω_8. Hence $t_h^{(i)} = 0$ and $u_h^{(i)} = 0$. Thus we have

$$f_i^2 = \alpha_{ii} f_i + \alpha_{ij} f_j + \alpha_{ik} f_k. \tag{9}$$

There exists a D_i in $\mathfrak{D}_i$ such that D_i is not in $\mathfrak{D}_0$ [4, p. 140]. For this D_i, $D_i e_j \neq 0$, for otherwise $D_i e_k = 0$ and D_i is in $\mathfrak{D}_0$. For this D_i we apply the derivation $D_i^{\tilde{S}} = SD_i S^{-1}$ to (9) and get $0 = (\alpha_{ij} - \alpha_{ik}) SD_i e_j$ or $\alpha_{ij} = \alpha_{ik}$. Equation (9) may be written as

$$f_i^2 = \alpha_i f_i + \beta_i f_j + \beta_i f_k. \tag{10}$$

Let x be an element of $\mathfrak{J}$. By computing x^2 and x^3 it may be seen that x satisfies the relation

$$x^3 + \alpha(x)x^2 + \beta(x)x + \gamma(x)1 = 0 \tag{11}$$

where $\alpha(x) = -(\xi_1 + \xi_2 + \xi_3)$, $\beta(x) = \sum \xi_i \xi_j - \sum n(c_i)$, and $\gamma(x) = \sum \xi_i n(c_i) - \xi_1 \xi_2 \xi_3 - \text{Sp}\,(c_1 c_2 c_3)$. Using the expressions given in [4] for $\text{Sp}\, x$, $\text{Sp}\, x^2$, and $\text{Sp}\, x^3$, we get $\alpha(x) = -\text{Sp}\, x$, $\beta(x) = \{(\text{Sp}\, x)^2 - \text{Sp}\, x^2\}/2$, and $\gamma(x) = \{3(\text{Sp}\, x)(\text{Sp}\, x^2) - (\text{Sp}\, x)^3 - 2\text{Sp}\, x^3\}/6$. We compute the coefficients in (11) for f_i. Equations (6) and (7) imply $\mathrm{Sp}\, f_i = \mathrm{Sp}\, e_i = 1$ and $\mathrm{Sp}\, f_i^2 = \mathrm{Sp}\, e_i^2 = \mathrm{Sp}\, e_i = 1$, (10) gives $\alpha_i + 2\beta_i = 1$, and (10) may be rewritten as

$$f_i^2 = \beta_i 1 + (1 - 3\beta_i) f_i. \tag{12}$$

Hence

$$\begin{aligned} f_i^3 &= \beta_i f_i + (1 - 3\beta_i) f_i^2 \\ &= (\beta_i - 3\beta_i^2) 1 + (1 - 5\beta_i + 9\beta_i^2) f_i. \end{aligned} \tag{13}$$

Moreover, $\mathrm{Sp}\, f_i^3 = 1 - 2\beta_i$. Substituting in (11),

$$f_i^3 - f_i^2 + (2/3)\beta_i 1 = 0. \tag{14}$$

Combining (12), (13), and (14) we get

$$((2/3\beta_i) - 3\beta_i^2) 1 + (9\beta_i^2 - 2\beta_i) f_i = 0,$$

but 1 and f_i being linearly independent, $9\beta_i^2 - 2\beta_i = 0$. Thus $\beta_i = 0$ or $\beta_i = 2/9$.

Sx was defined as $S(\gamma 1 + x_0) = \gamma 1 + \sigma^{-1/2} S_1 x_0$ for $x = \gamma 1 + x_0$. Let $S'x = \gamma 1 - \sigma^{-1/2} S_1 x_0$. Then S' has all the properties we have derived for S. $\mathrm{Sp}\ x = \mathrm{Sp}\ (Sx) = 3\gamma$ and $Sx + S'x = 2\gamma 1$ imply $Sx + S'x = (2/3)(\mathrm{Sp}\ x)1$. Calling $S'e_i = f_i'$, we have $f_i' = (2/3)1 - f_i$, and

$$\begin{aligned} (f_i')^2 = \beta_i' 1 + (1 - 3\beta_i') f_i' &= (-\beta_i' + 2/3) 1 + (-1 + 3\beta_i') f_i \\ = ((2/3)1 - f_i)^2 &= (\beta_i + 4/9) 1 + (-3\beta_i - 1/3) f_i. \end{aligned}$$

Comparing coefficients we get $\beta_i' = 2/9 - \beta_i$. Thus if $\beta_i = 2/9$, $\beta_i' = 0$; and if $\beta_i = 0$, $\beta_i' = 2/9$. Therefore (by replacing S by S', if necessary) we may assume either that all $\beta_i = 0$ or that exactly one $\beta_i = 0$ (and $\beta_j = \beta_k = 2/9$). We shall show that the second case leads to a contradiction.

For any t_i in $\mathfrak{T}_i$ there is a D_i in $\mathfrak{D}_i$ such that $D_i e_j = t_i$. Then for this D_i, $SD_iS^{-1} f_j = St_i$. Apply SD_iS^{-1} to the equation obtained from (12) by replacing i by j:

$$2 f_j u_i = (1 - 3\beta_j) u_i. \tag{15}$$

Since $1u_i = (f_i + f_j + f_k) u_i$, if we use (15) together with (15) in which j is replaced by k we have

$$f_i u_i = (3\beta_j + 3\beta_k) u_i / 2. \tag{16}$$

If $\beta_i = 0$ and $\beta_j = \beta_k = 2/9$, f_i is an idempotent and $f_i u_i = (2/3) u_i$, but

[1, p. 550] the only characteristic roots of R_{f_i} are 0, 1/2, and 1. Since we may choose $u_i = St_i \neq 0$, we have a contradiction. Thus $\beta_i = 0$ for all i, and the elements f_i are idempotents in $\mathfrak{J}$. Since $(f_i+f_j)^2 = (1-f_k)^2$, we have $f_if_j = 0$, or the f_i are pairwise orthogonal.

Since $\beta_j + \beta_k = 0$, equations (15) and (16) imply that $f_ju_i = u_i/2$, $f_iu_i = 0$ for any $u_i = St_i$ in $S\mathfrak{T}_i$. Hence

$$(17)\qquad \begin{aligned} (Se_i)^2 &= S(e_i^2) = Se_i, & (Se_i)(Se_j) &= S(e_ie_j) = 0, \\ (Se_i)(St_i) &= S(e_it_i) = 0, & (Se_j)(St_i) &= S(e_jt_i) = St_i/2. \end{aligned}$$

In order to show that S is an automorphism of $\mathfrak{J}$, it remains only to show that

$$(18)\qquad (St_i)(St_i') = S(t_it_i'), \qquad (St_i)(St_j) = S(t_it_j).$$

We compute the product t_it_i' in $\Omega(e_j+e_k)$ for any t_i and t_i' in $\mathfrak{T}_i$ as follows: there is a D_i in $\mathfrak{D}_i$ such that $D_ie_j = t_i$. Also [4, p. 140]

$$(19)\qquad D_it_i' = \theta(e_j - e_k) + t_i \qquad \text{for } \theta \in \Omega,\ t_i{}' \in \mathfrak{T}_i.$$

Apply this D_i to $e_jt_i' = t_i'/2$ to obtain

$$(20)\qquad t_it_i' = -\theta(e_j + e_k)/2$$

for θ in (19). To compute the product $(St_i)(St_i')$ we apply the derivation SD_iS^{-1} to $(Se_j)(St_i') = St_i'/2$. Using (17), we obtain

$$(21)\qquad (St_i)(St_i') = -\theta(Se_j + Se_k)/2$$

for θ in (19). Hence the first of equations (18) holds. Finally, given t_i in $\mathfrak{T}_i$, t_j in $\mathfrak{T}_j$, apply the derivation D_i in $\mathfrak{D}_i$ satisfying $D_ie_j = t_i$ to the equation $e_jt_j = 0$. Since

$$(22)\qquad D_it_j = t_j' + t_k, \qquad t_j' \in \mathfrak{T}_j,\ t_k \in \mathfrak{T}_k$$

[4, p. 140], this gives $t_it_j = -t_k/2$ for t_k in (22). To compute the product $(St_i)(St_j)$ we apply SD_iS^{-1} to $(Se_j)(St_j) = 0$ in (17) to obtain $(St_i)(St_j) = -St_k/2$ for t_k in (22). Hence $(St_i)(St_j) = S(t_it_j)$.

It has been shown that S is an automorphism of $\mathfrak{J}$; it remains to show that it is unique. Let R be an automorphism of $\mathfrak{J}$ such that $D \rightarrow D^{\mathfrak{J}} = SDS^{-1} = RDR^{-1}$. In §1 we saw that $R\mathfrak{J}_0 = \mathfrak{J}_0$. Thus on $\mathfrak{J}_0$, $R^{-1}S$ commutes with all D. Hence $R^{-1}S = \sigma I$, $\sigma \neq 0$ in Ω, on $\mathfrak{J}_0$; that is, $S = \sigma R$ on $\mathfrak{J}_0$. Choose t_1 in $\mathfrak{T}_1$, t_2 in $\mathfrak{T}_2$ such that $t_1t_2 = t_3 \neq 0$ in $\mathfrak{T}_3$. $St_1 = \sigma Rt_1$, $St_2 = \sigma Rt_2$, and $St_3 = (St_1)(St_2) = \sigma^2(Rt_1)(Rt_2) = \sigma^2 Rt_3$. Hence $\sigma = 1$. Since $R1 = S1$, $R = S$ on $\mathfrak{J}$.

4. Lie algebras of type F. Here we assume merely that the base field Φ is of characteristic 0. A Lie algebra $\mathfrak{L}$ is said to be of *type* F if

$\mathfrak{L}_\Omega$ is the Lie algebra F_4 over Ω where Ω is the algebraic closure of Φ. Our determination of the Lie algebras of type F is given in terms of the exceptional central simple Jordan algebras over Φ defined in §1.

In [6] Jacobson characterizes Lie algebras of type G as the derivation algebras of Cayley algebras over Φ. The next three theorems in this paper are restatements of analogous theorems for algebras of type G. The statements made in §1 about exceptional simple Jordan algebras together with Theorem 1 allow us to use Jacobson's proofs. We shall merely give a brief outline of the proofs of Theorems 2 and 4.

THEOREM 2. *Let $\mathfrak{J}_1$ and $\mathfrak{J}_2$ be exceptional central simple Jordan algebras over a field Φ of characteristic* 0 *such that $\mathfrak{D}(\mathfrak{J}_1)\cong\mathfrak{D}(\mathfrak{J}_2)$. Then there exists a unique isomorphism S between $\mathfrak{J}_1$ and $\mathfrak{J}_2$ such that the given isomorphism between $\mathfrak{D}(\mathfrak{J}_1)$ and $\mathfrak{D}(\mathfrak{J}_2)$ has the form $D\to E=SDS^{-1}$.*

Let Ω be the algebraic closure of Φ. The algebras $\mathfrak{J}_1$ and $\mathfrak{J}_2$ may be regarded as subrings of $\mathfrak{J}$, the unique exceptional simple Jordan algebra over Ω. The isomorphism between $\mathfrak{D}(\mathfrak{J}_1)$ and $\mathfrak{D}(\mathfrak{J}_2)$ may be extended to an automorphism of $\mathfrak{D}(\mathfrak{J})$. By our lemma there is a linear transformation S_1 of $\mathfrak{J}_0$ such that the given mapping of $\mathfrak{D}$ has the form $D\to S_1DS_1^{-1}$ on $\mathfrak{J}_0$. Using bases of $(\mathfrak{J}_1)_0$ and $(\mathfrak{J}_2)_0$ as bases of $\mathfrak{J}_0$, the matrix of S_1 may be taken with elements in Φ and moreover S_1 maps $(\mathfrak{J}_1)_0$ onto $(\mathfrak{J}_2)_0$. By the proof of Theorem 1 there is a unique automorphism S of $\mathfrak{J}$ which maps $\mathfrak{J}_1$ onto $\mathfrak{J}_2$ and such that the isomorphism between $\mathfrak{D}(\mathfrak{J}_1)$ and $\mathfrak{D}(\mathfrak{J}_2)$ is given by $D\to SDS^{-1}$.

THEOREM 3. *If $\mathfrak{J}$ is an exceptional central simple Jordan algebra over a field Φ of characteristic* 0, *then the group of automorphisms of $\mathfrak{D}(\mathfrak{J})$ is isomorphic to the group of automorphisms of $\mathfrak{J}$.*

THEOREM 4. *A necessary and sufficient condition that a Lie algebra $\mathfrak{L}$ over a field Φ of characteristic* 0 *be of type* F *is that $\mathfrak{L}\cong\mathfrak{D}(\mathfrak{J})$, $\mathfrak{J}$ an exceptional central simple Jordan algebra over Φ.*

Let $\mathfrak{L}$ be a Lie algebra of type F over Φ and $\mathfrak{J}$ the unique exceptional Jordan algebra over Ω, the algebraic closure of Φ. The basal elements of $\mathfrak{L}$ may be represented as derivations D_k of $\mathfrak{J}$. If (e_i) is a basis of $\mathfrak{J}$ and $D_ke_i=\sum\gamma_{ij}^{(k)}e_j$, the $\gamma_{ij}^{(k)}$ may be taken as elements in a finite Galois extension P of Φ such that $\mathfrak{L}_P$ is isomorphic to the derivation algebra of (e_i) over P. Using Theorem 2 it may be shown that there is a (1-1) representation of the Galois group of P over Φ by semi-linear transformations of (e_i) over P which commute with the D_k. Thus the conditions of the lemma of [6, p. 782] are satisfied and the set of elements invariant under these semi-lineer transformations is a vector space of dimension 27 over Φ. This space is closed with respect to multiplication and is a central simple Jordan algebra over Φ. The D_k are derivations of this algebra and the theorem is proved.

REMARK. As a corollary to Theorems 2 and 4 we may remove the assumption of algebraic closure in Theorem 1: if $D \rightarrow D^{\mathfrak{S}}$ is an automorphism of $\mathfrak{L} = \mathfrak{D}(\mathfrak{J})$, any Lie algebra of type F over Φ of characteristic 0, then there is a unique automorphism S of $\mathfrak{J}$ such that $D^{\mathfrak{S}} = SDS^{-1}$.

THEOREM 5. *A Lie algebra* $\mathfrak{L}$ *over a field* Φ *of characteristic* 0 *is simple with multiplication center* P *and of type* F *over* P *if and only if* $\mathfrak{L} = \mathfrak{D}(\mathfrak{A})$ *for some exceptional simple Jordan algebra* $\mathfrak{A}$ *with center* P.

If $\mathfrak{A}$ is an exceptional simple Jordan algebra over Φ with center P, then $\mathfrak{A}$ is central simple over P. Since Φ is of characteristic 0, the elements of P are such that $D\mathrm{P} = 0$ for all D in $\mathfrak{D}(\mathfrak{A})$ [5]. Thus $\mathfrak{D}(\mathfrak{A})$ may be regarded as an algebra over P, since $D(\rho x) = \rho(Dx)$ for ρ in P, x in $\mathfrak{A}$. Therefore $\mathfrak{D}(\mathfrak{A})$ over $\mathrm{P} = \mathfrak{D}(\mathfrak{A}$ over P) or $\mathfrak{D}(\mathfrak{A})$ is a Lie algebra of type F over P.

Conversely, by Theorem 4, $\mathfrak{L}$ over $\mathrm{P} \cong \mathfrak{D}(\mathfrak{A}$ over P) where $\mathfrak{A}$ over P is an exceptional central simple Jordan algebra. Over Φ, $\mathfrak{A}$ is an exceptional simple Jordan algebra. Thus $\mathfrak{L} \cong \mathfrak{D}(\mathfrak{A})$, since $\mathfrak{D}\mathrm{P} = 0$.

5. **Lie algebras of type F over a real closed field.** Let Φ be a real closed field. It is known [3] that there are three nonisomorphic Lie algebras of type F over Φ. Then our Theorems 2 and 4 imply that there are three non-isomorphic exceptional central simple Jordan algebras $\mathfrak{J} = \mathfrak{H}(\mathfrak{C}, p)$ over Φ.

Let $\mathfrak{C}_0$ be the Cayley algebra with divisors of zero over Φ, $\mathfrak{C}_1$ be the Cayley division algebra over Φ, and

$$p_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \in \Phi_3.$$

THEOREM 6. *The three nonisomorphic exceptional central simple Jordan algebras over a real closed field* Φ *are*

$$\mathfrak{H}(\mathfrak{C}_0, 1), \qquad \mathfrak{H}(\mathfrak{C}_1, 1), \qquad \mathfrak{H}(\mathfrak{C}_1, p_0) \tag{23}$$

and the three Lie algebras of type F *over* Φ *are their derivation algebras.*

For Jacobson has recently proved[3] that $\mathfrak{H}(\mathfrak{C}_0, p) \cong \mathfrak{H}(\mathfrak{C}_0, 1)$ for any

[3] In a letter to Schafer dated 2/17/52 Jacobson remarks that he has proved $\mathfrak{H}(\mathfrak{C}_0, p) \cong \mathfrak{H}(\mathfrak{C}_0, 1)$ for any p where $\mathfrak{C}_0$ is the unique Cayley algebra with divisors of zero over an arbitrary Φ.

p. Let $\mathfrak{C}$ be any Cayley algebra over an arbitrary Φ. Let $\tilde{p}=\alpha g p g'$ for $\alpha\neq 0$ in Φ, g nonsingular in Φ_3; that is, $\tilde{p}$ differs from a matrix congruent to p by only a nonzero scalar factor. Then it is easy to see that $\mathfrak{H}(\mathfrak{C}, p)\cong\mathfrak{H}(\mathfrak{C}, \tilde{p})$ under the mapping $x\rightarrow gxg^{-1}$. Over a real closed field Φ, any p is congruent to one of

$$1, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad -1.$$

Hence any $\mathfrak{J}=\mathfrak{H}(\mathfrak{C}, p)$ over a real closed field Φ is isomorphic to either $\mathfrak{H}(\mathfrak{C}, 1)$ or $\mathfrak{H}(\mathfrak{C}, p_0)$. Combining these results we see that $\mathfrak{J}$ is isomorphic to one of the algebras (23). But three nonisomorphic algebras do exist, and so these are the algebras (23).

References

1. A. A. Albert, *A structure theory for Jordan algebras*, Ann. of Math. vol. 48 (1947) pp. 546–567.

2. E. Cartan, *Les groupes projectifs qui ne laissent invariante aucune multiplicité plane*, Bull. Soc. Math. France vol. 41 (1913) pp. 53–96.

3. ———, *Les groupes reels simples et continus*, Ann. École Norm. vol. 31 (1914) pp. 263–355.

4. C. Chevalley and R. D. Schafer, *The exceptional simple Lie algebras F_4 and E_6*, Proc. Nat. Acad. Sci. U. S. A. vol. 36 (1950) pp. 137–141.

5. N. Jacobson, *Abstract derivation and Lie algebras*, Trans. Amer. Math. Soc. vol. 42 (1937) pp. 206–224.

6. ———, *Cayley numbers and normal simple Lie algebras of type* G, Duke Math. J. vol. 5 (1939) pp. 775–783.

7. ———, *Derivation algebras and multiplication algebras of semisimple Jordan algebras*, Ann. of Math. vol. 50 (1949) pp. 866–874.

8. R. D. Schafer, *The exceptional simple Jordan algebras*, Amer. J. Math. vol. 70 (1948) pp. 82–94.

9. ———, *A theorem on the derivations of Jordan algebras*, Proc. Amer. Math. Soc. vol. 2 (1951) pp. 290–294.

University of Pennsylvania