A SOLUTION OF THE SINGULAR INITIAL VALUE PROBLEM FOR THE EULER-POISSON-DARBOUX EQUATION[1]

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1. Introduction. This paper is concerned with the solution of the Cauchy problem

$$\Delta u = u_{tt} + \frac{k}{t} u_t \qquad \left(\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_m^2}\right), \; t > 0; \tag{1}$$

$$u(x_1, \cdots, x_m, 0) = f(x_1, \cdots, x_m), \qquad u_t(x_1, \cdots, x_m, 0) = 0, \tag{2}$$

where k is a (real or complex) parameter.

Equation (1), for special values of k and m, occurs in many important and classical problems since the time of Euler. Euler [10][2] considered $m=1$ and a partial differential equation (denoted by $E(\beta, \beta')$ by Darboux [8]) which is equivalent to (1) when $\beta=\beta' =k/2$. The equation (1) with $m=1$ was later treated by Poisson [17]. An exposition of the theory of Euler and Poisson is given by Darboux [8]. These treatments were not concerned with the singular initial values (2). The important special case $m=1$, $k=1/3$, of (1), (2), plays an important rôle in the work of Tricomi [22]. Poisson [18], in solving the equation of the propagation of sound waves in three-dimensional space, considered the case $m=3$, $k=2$ in (1). Asgeirsson [1] gave a solution of (1), (2) for all positive integers m and $k=m-1$. Related questions were treated by John [12; 13]. Equation (1), for $m=1$, $k=-1, -2, -3, \cdots$, appears in the work of Martin [16] and Diaz and Martin [9]. Kapilevic [14] has given solutions of (1), (2) for $m=1$, 2 and $0<k<1$. The most frequently discussed special case of (1) is, of course, $k=0$, the wave equation. Among the classical accounts one finds Volterra [23], Tedone [21], and Hadamard [11], while among the more recent papers one might mention Bureau [3; 4], Diaz and Martin [9], Linés [15], and Riesz [20].

All these various cases were treated by special methods. A unified solution of (1), (2) for all real values of k was recently given by Weinstein [26], by a combination of a generalized method of descent with a recurrence formula. For the cases $k=-1, -3, \cdots$, Wein-

Presented to the Society, October 25, 1952; received by the editors August 18, 1952 and, in revised form, February 13, 1953.

[1] This paper was sponsored by the Office of Naval Research.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

stein assumed that $u(x_i, t)$ satisfies certain differentiability conditions with respect to t at $t=0$. Under these conditions, he found that a solution exists only if the initial value function $f(x_i)$ is a polyharmonic function of order $(1-k)/2$.

The present paper contains another solution of the problem (1), (2), for all k. In §6 a solution for arbitrary $f(x_i)$ is found for the exceptional values $k=-1, -3, \cdots$. The t derivative of order $1-k$ of this solution is logarithmic at $t=0$ when $f(x_i)$ is not polyharmonic of order $(1-k)/2$.[3]

The following method of solution is employed in this paper. Hadamard's "method of descent" is used to obtain solutions of (1), (2) for $k=m, m+1, m+2, \cdots$ from the known solution for $k=m-1$. It is then directly verified that the resulting formula gives a solution of the problem for any k with Re $k>m-1$. This part is in common with Weinstein's procedure [27]. However, the definite integral, in terms of which the solution is expressed for Re $k>m-1$, is divergent for Re $k\leqq m-1$. In §4 this integral is continued analytically in k and it is verified that the resulting formula does indeed furnish a solution of (1), (2) for Re $k\leqq m-1$, with the exception of $k=-1, -3, \cdots$. To avoid misunderstanding, it should be noted that the method used in this paper differs from that used by M. Riesz [19; 20] for solving *regular* Cauchy problems, although both methods employ analytic continuation of definite integrals. As is well known, M. Riesz uses a modified fundamental solution depending on a parameter α. He obtains the solution by analytic continuation of a Green's identity with respect to α. In the present method the family of differential equations (1) depending on a parameter k is considered, and the solution is obtained by analytic continuation in k. Neither a fundamental solution nor a Green's identity is used. It is true that both methods involve the analytic continuation of integrals which are related to the Riemann-Liouville integral. However, it should be remembered that the analytic continuation of divergent integrals, based on similar principles, but for different purposes, occurs already in Cauchy [6] (see also Weierstrass [24] and Whittaker and Watson [28, p. 243]).

2. **The mean value function** M. Let $f(x_1, \cdots, x_m)=f(x_i)$ possess continuous derivatives of second order. Define the "mean value function" M by the equation

$$(3) \qquad M(x_i, r; f) = \frac{1}{\omega_m}\int_{\sum_{j=1}^m \beta_j^2=1} f(x_i+\beta_i r)\,d\omega_m, \qquad r \geqq 0,$$

[3] Our attention was drawn to this behavior by the example $u=x^2+t^2\log t$ for $k=-1$, $m=1$, kindly communicated by Professor B. Friedman in a letter.

where $d\omega_m$ is the element of area of the m-dimensional unit sphere, and ω_m is the surface area of this sphere ($\omega_m = 2\pi^{m/2}/\Gamma(m/2)$). $M(x_i, r; f)$ is the mean value of the function f over the surface of the m-dimensional sphere with center at $(x_1, \cdots, x_m)$ and radius r. When $m=1$, it is understood that the mean value integral in (1) is to be replaced by

$$\frac{f(x+r)+f(x-r)}{2}.$$

It is well known that $M(x_i, r; f)$ satisfies the so-called Darboux (sometimes also referred to as Euler-Poisson) equation [1; 7]

$$\Delta M = M_{rr} + \frac{m-1}{r} M_r, \qquad \left(\Delta = \sum_{j=1}^{m} \frac{\partial^2}{\partial x_j^2}\right), \ r > 0, \tag{4}$$

together with the initial conditions

$$\begin{aligned} M(x_i, 0) &= f(x_i), \\ M_r(x_i, 0) &= 0. \end{aligned} \tag{5}$$

More precisely, M is twice continuously differentiable and satisfies (4) for $r>0$, has continuous first derivatives with respect to all variables for $r \geqq 0$, and

$$\begin{aligned} &\lim_{r\to+0,\, z_i\to x_i} M(z_i, r; f) = f(x_1, \cdots, x_m), \\ &\lim_{r\to+0,\, z_i\to x_i} M_r(z_i, r; f) \\ &\quad = \lim_{r\to+0,\, z_i\to x_i} \frac{1}{\omega_m} \int_{\sum_{j=1}^m \beta_j^2 = 1} \left\{ \sum_{k=1}^{m} \beta_k \frac{\partial f}{\partial x_k}(z_i + \beta_i r) \right\} d\omega_m = 0. \end{aligned} \tag{5'}$$

It will now be shown that an appropriate transformation of $M(x_i, r; f)$ yields a solution of the more general initial value problem (1), (2).

3. **Solution of the initial value problem for** Re $k > m-1$. When $k-(m-1)$ is a positive integer, the initial value problem (1), (2) can be solved by Hadamard's method of descent [11]; that is, by noticing that this problem coincides with the initial value problem (1), (2) in a space of $k+1$ dimensions $x_1, \cdots, x_m, \cdots, x_{k+1}$, where the initial value function f depends only on the variables $x_1, \cdots, x_m$. The solution, which is the mean value of the initial value function over a $(k+1)$-dimensional sphere, reduces to a volume integral over an m-dimensional sphere, namely

$$u(x_i, r) = \frac{\omega_{k+1-m}}{\omega_{k+1}} \int_{\sum_{j=1}^{m} \beta_j^2 \leq 1} f(x_1 + \beta_1 r, \cdots, x_m + \beta_m r) \cdot (1 - \beta_1^2 - \cdots - \beta_m^2)^{(k-m-1)/2} d\beta_1 \cdots d\beta_m.$$

Using the definition (1) of M, this formula may be written[4]

$$u(x_i, t) = \frac{2\Gamma((k+1)/2)}{\Gamma((k+1-m)/2)\Gamma(m/2)} \cdot \int_0^1 M(x_i, \alpha t; f)(1 - \alpha^2)^{(k-m-1)/2} \alpha^{m-1} d\alpha. \tag{6}$$

Since f is twice continuously differentiable, (6) may be differentiated under the integral sign, to obtain

$$\Delta u - u_{tt} - \frac{k}{t} u_t = \frac{2\Gamma((k+1)/2)}{\Gamma((k+1-m)/2)\Gamma(m/2)} \cdot \int_0^1 \left[\Delta M - \alpha^2 M_{rr} - \frac{k}{t} \alpha M_r \right] (1 - \alpha^2)^{(k-m-1)/2} \alpha^{m-1} d\alpha. \tag{7}$$

However, from (4) it follows that

$$\Delta M(x_i, \alpha t; f) = M_{rr} + ((m-1)/\alpha t) M_r, \tag{8}$$

so that

$$\begin{aligned} \Delta u - u_{tt} - \frac{k}{t} u_t &= \frac{2\Gamma((k+1)/2)}{\Gamma((k+1-m)/2)\Gamma(m/2)} \cdot \int_0^1 \left[(1 - \alpha^2) M_{rr} + \frac{m - 1 - k\alpha^2}{\alpha t} M_r \right] \cdot (1 - \alpha^2)^{(k-m-1)/2} \alpha^{m-1} d\alpha, \\ &= \frac{2\Gamma((k+1)/2)}{\Gamma((k+1-m)/2)\Gamma(m/2)} \cdot \int_0^1 \frac{d}{d\alpha} \left\{ \frac{M_r}{t} (1 - \alpha^2)^{(k-m+1)/2} \alpha^{m-1} \right\} d\alpha, \\ &= 0, \end{aligned} \tag{9}$$

for Re $k > m-1$. Thus u is a solution of (1) for $t > 0$. That u also satisfies the initial conditions (2) follows easily from (5′) and

[4] (6) may be written $u(x_i, t) = (\Gamma((k+1)/2)/\Gamma(m/2)) t^{1-k} I^{(k-m+1)/2}[M(x_i, \beta^{1/2}; f)\beta^{(m-2)/2}](t^2)$, as pointed out by the referee, where $I^a F(t) = (1/\Gamma(a)) \int_0^t (t-\beta)^{a-1} \cdot F(\beta) d\beta$ is the Riemann-Liouville integral, by putting $\beta = \alpha^2 t^2$.

$$\int_0^1 (1-\alpha^2)^{(k-m-1)/2}\alpha^{m-1}d\alpha = \frac{\Gamma((k+1-m)/2)\Gamma(m/2)}{2\Gamma((k+1)/2)}.$$

4. **Solution of the initial value problem for** Re $k \leqq m-1$. If, for some $(x_1, \cdots, x_m, t)$, it happens that $M(x_i, t; f) \neq 0$, then the integral in the definition (6) of $u(x_i, t)$ does not converge for Re $k \leqq m-1$. In this case, $u(x_i, t)$ will be defined by analytic continuation in k. Let p be a non-negative integer and suppose that for $r>0$ the $m+3$ functions $M(x_i, r; f)$, M_r, M_{rr}, and $M_{x_ix_i}$ have $p+1$ continuous partial derivatives with respect to r (a simple sufficient condition for this is that f possess continuous partial derivatives of all orders up to $p+3$). Then, for fixed x_i and t, the function

$$\begin{aligned} N_p(x_i, \alpha, t; f) \\ &= M(x_i, \alpha t; f) - \sum_{n=0}^{p} \frac{(\alpha-1)^n}{n!}\frac{\partial^n}{\partial\alpha^n} M(x_i, \alpha t; f)\bigg|_{\alpha=1} \\ &= M(x_i, \alpha t; f) - \sum_{n=0}^{p} \frac{t^n(\alpha-1)^n}{n!}\frac{\partial^n}{\partial t^n} M(x_i, t; f) \end{aligned} \tag{10}$$

is of the order $(\alpha-1)^{p+1}$ near $\alpha=1$, and a similar assertion holds for ΔN_p, $\partial N_p/\partial t$, and $\partial^2 N_p/\partial t^2$.

Equation (6) may now be written

$$\begin{aligned} u(x_i, t) &= \frac{2\Gamma((k+1)/2)}{\Gamma((k+1-m)/2)\Gamma(m/2)} \\ &\cdot \left\{ \int_0^1 N_p(x_i, \alpha, t; f)(1-\alpha^2)^{(k-m-1)/2}\alpha^{m-1}d\alpha \right. \\ &+ \int_0^1 \sum_{n=0}^{p} \frac{t^n(\alpha-1)^n}{n!}\frac{\partial^n}{\partial t^n} M(x_i, t; f) \\ &\left. \cdot (1-\alpha^2)^{(k-m-1)/2}\alpha^{m-1}d\alpha \right\}, \end{aligned} \tag{11}$$

for Re $k>m-1$. By virtue of the identity

$$\begin{aligned} &\int_0^1 \alpha^{m-1}(\alpha-1)^n(1-\alpha^2)^{(k-m-1)/2}d\alpha \\ &= \sum_{l=0}^{n} (-1)^{n+l}\binom{n}{l}\int_0^1 \alpha^{m+l-1}(1-\alpha^2)^{(k-m-1)/2}d\alpha \\ &= \frac{1}{2}\Gamma\left(\frac{k+1-m}{2}\right)(-1)^n\sum_{l=0}^{n}(-1)^l\binom{n}{l}\frac{\Gamma((m+l)/2)}{\Gamma((k+l+1)/2)}, \end{aligned}$$

one can write

$$
\begin{aligned}
u(x_i, t) = {} & \frac{2\Gamma((k+1)/2)}{\Gamma((k+1-m)/2)\Gamma(m/2)} \\
& \cdot \int_0^1 N_p(x_i, \alpha, t; f)(1-\alpha^2)^{(k-m-1)/2}\alpha^{m-1}d\alpha \\
& + \frac{\Gamma((k+1)/2)}{\Gamma(m/2)} \sum_{n=0}^{p} \frac{(-t)^n}{n!} \frac{\partial^n M}{\partial t^n}(x_i, t; f) \\
& \cdot \sum_{l=0}^{n} (-1)^l \binom{n}{l} \frac{\Gamma((m+l)/2)}{\Gamma((k+l+1)/2)}.
\end{aligned} \tag{12}
$$

The integral in this formula, and its first two partial derivatives with respect to x_i and t (which may be taken under the integral sign), all converge for Re $k>m-2p-3$, while the sum in (12) is an entire function of k. Thus (12) yields an analytic continuation of (6). Moreover,

$$\frac{1}{\Gamma((k+1)/2)}\left[\Delta u - u_{tt} - \frac{k}{t}u_t\right]$$

is, for fixed x_i and t, an analytic function of k for Re $k>m-2p-3$ and vanishes for Re $k>m-1$. Hence it vanishes for all Re $k>m-2p-3$. Thus, (12) is a solution of (1) for all such values of k except (possibly) negative odd integers.

In order to verify that (12) satisfies the initial conditions (2), it is assumed further that, given any point $(x_1, \cdots, x_m, 0)$, the functions M, $\partial M/\partial r, \cdots, \partial^{p+2}M/\partial r^{p+2}$, which exist for $r>0$, are all bounded in absolute value in a sufficiently small "neighborhood" of this point. (It is remarked that this additional hypothesis is automatically fulfilled if f satisfies the simple sufficient condition mentioned above that f have continuous partial derivatives of all orders up to $p+3$, since, from the definition (3) of M, the successive derivatives of M with respect to r may be obtained by differentiating f under the integral sign in (3)). If $p\geqq 1$, (12) satisfies the initial conditions (2), since N_p is of order t^{p+1} as $t\to 0$, $\partial N_p/\partial t$ is of order t^p as $t\to 0$, and M satisfies (5'). If $p=0$, the initial conditions are still satisfied. For the term of order t^0 in the integral for $\partial u/\partial t$ has for a coefficient $\partial M/\partial t$, which approaches zero with t.

By a simple extension of an argument of Zaremba [29] (see Courant-Hilbert [7, pp. 381–382]) one finds that the solution of the initial value problem (1), (2) is unique for Re $k\geqq 0$. That is, there is at most one function u, twice continuously differentiable for $t>0$

and once continuously differentiable for $t \geqq 0$, satisfying (1), (2). When Re $k<0$, the solution is not unique, as pointed out by Weinstein [26; 27]. In fact, one may add to u any function of the form $t^{1-k}v(x_i, t)$ where $v(x_i, t)$ is a regular solution of equation (1) with k replaced by $2-k$.

Incidentally, for $m=2$, $k=0$, taking $p=0$ in (12) yields a result in agreement with equation (60) of page 278 of Hadamard [11].

A simple sufficient condition for the validity of the solution (12) of the initial value problem (1), (2) is that f have $p+3$ continuous derivatives and that Re $k>m-2p-3$. From this it follows that f need only have a number of continuous derivatives greater than $(m+3-\text{Re } k)/2$. In the special case when $m-k$ is a positive odd integer, the integral term in (12) disappears (see §6) and it is necessary only for f to have exactly $(m+3-k)/2$ continuous derivatives. When Re $k>m-1$, no analytic continuation is used, and f need only have two continuous derivatives. For real k, these results agree with those of Weinstein [27].

To solve the initial value problem

$$u(x_i, 0) = 0, \qquad u_t(x_i, 0) = g(x_i), \tag{13}$$

for the wave equation ($k=0$), we note that if the function $v(x_i, t)$ satisfies the differential equation (1) with $k=2$ and has the boundary value $g(x_i)$, then (see Weinstein [26]) $u(x_i, t)=tv(x_i, t)$. Thus, one can solve the Cauchy problem

$$w(x_i, 0) = f(x_i), \qquad w_t(x_i, 0) = g(x_i), \tag{14}$$

for the wave equation $\Delta w = w_{tt}$ by means of formula (12). For this solution to be valid, it is sufficient that $f(x_i)$ and $g(x_i)$ have continuous derivatives of all orders up to $[m/2]+2$ and $[m/2]+1$ respectively, for $m \geqq 2$. Here, $[m/2]$ denotes the greatest integer $\leqq m/2$. For $m=1$, both f and g must be required to have two continuous derivatives. For $m \geqq 2$ the condition on g agrees with that given by Courant-Hilbert [7, p. 399]. The conditions on f given by Courant-Hilbert [7, p. 399 and p. 469] are insufficient, as already pointed out by Weinstein [27].

5. **The odd, negative integers.** When k approaches an odd negative integer, the solution (12) becomes infinite because of the factor $\Gamma((k+1)/2)$. As has already been pointed out, any function of the form $t^{1-k}v(x_i, t)$, with v a regular solution of equation (1) with k replaced by $2-k$, may be added to (12) when $k<0$. The function will now be determined in such a way that the solution $u(x_i, t)+t^{1-k}v(x_i, t)$

remains a solution of the problem (1), (2) when k becomes a particular odd negative integer $-(2r+1)$. In formula (12), take $p \geqq 2r+3$ as well as $p \geqq (m-3+(2r+2))/2$ and suppose that $M(x_i, t; f)$ has at least $2r+4$ continuous derivatives. Then the integral in (12) is of the order t^{2r+4} as $t \to 0$ and (12) gives a solution of (1), (2) for k in the neighborhood of $-(2r+1)$. Rewriting the finite sum in (12) by expanding M and its derivatives in a Taylor series about $t=0$, one obtains

$$\text{(15)} \quad \begin{aligned} u(x_i, t) = {} & \frac{\Gamma((k+1)/2)}{\Gamma(m/2)} \sum_{n=0}^{p} \frac{t^n}{n!} \frac{\partial^n M(x_i, 0; f)}{\partial t^n} \frac{\Gamma((m+n)/2)}{\Gamma((k+n+1)/2)} \\ & + O(t^{2r+4}). \end{aligned}$$

It is easily seen that all odd derivatives of M vanish at $t=0$, so that one need only sum over even n. From (15)

$$\text{(16)} \quad \begin{aligned} & \lim_{k \to -(2r+1)} \frac{1}{\Gamma((k+1)/2)} u(x_i, t) \\ & = \frac{t^{2r+2}}{(2r+2)!\Gamma(m/2)} \frac{\partial^{2r+2} M(x_i, 0; f)}{\partial t^{2r+2}} \frac{\Gamma((m+2r+2)/2)}{\Gamma(1)} \\ & \quad + O(t^{2r+4}). \end{aligned}$$

The function $[\Gamma((k+1)/2)]^{-1} u(x_i, t)$ is regular and satisfies the differential equation (1) for all Re $k > m-2p-3$ and in particular for $k = -(2r+1)$. Therefore $v(x_i, t) = t^{-(2r+2)} \lim [\Gamma((k+1)/2)]^{-1} u(x_i, t)$ satisfies the differential equation (1) with k replaced by $2-[-(2r+1)] = 2r+3$ and has the initial values

$$\text{(17)} \quad \begin{aligned} v(x_i, 0) &= \frac{\Gamma((m+2r+2)/2)}{(2r+2)!\Gamma(m/2)} \frac{\partial^{2r+2} M(x_i, 0; f)}{\partial t^{2r+2}}, \\ v_t(x_i, 0) &= 0. \end{aligned}$$

It is easily shown by the use of Green's theorem (see Courant and Hilbert [7, p. 261]) that

$$\frac{\partial^{2r+2} M(x_i, 0; f)}{\partial t^{2r+2}} = \frac{\Gamma(m/2)(2r+2)!}{2^{2r+2}(r+1)!\Gamma((m+2r+2)/2)} \Delta^{r+1} f,$$

where $\Delta^{r+1} f$ is the $(r+1)$st iterated Laplacian of f. Hence, (17) becomes

$$\text{(18)} \qquad v(x_i, 0) = \frac{1}{(r+1)!2^{2r+2}} \Delta^{r+1} f, \qquad v_t(x_i, 0) = 0.$$

Let $w(x_i, t)$ be the (unique for Re $k \leqq 2$) solution of the initial value problem

$$\Delta w = w_{tt} + \frac{2-k}{t} w_t,$$

(19)
$$w(x_i, 0) = \frac{1}{(r+1)!2^{2r+2}} \Delta^{r+1} f,$$

$$w_t(x_i, 0) = 0.$$

w is an analytic function of k which coincides with v for $k = -(2r+1)$. Thus $[\Gamma((k+1)/2)]^{-1} u(x_i, t) - t^{1-k} w(x_i, t)$ is an analytic function of k satisfying the differential equation (1) and having a zero at $k = -(2r+1)$. Hence $\Gamma((k+1)/2)$ times it is regular at this point. That is, $u(x_i, t) - \Gamma((k+1)/2) t^{1-k} w(x_i, t)$ is analytic in k at $k = -(2r+1)$. It satisfies the boundary value problem (1), (2) in the neighborhood of this value and hence also at $k = -(2r+1)$. A solution of the boundary value problem (1), (2) at $k = -(2r+1)$ is thus

$$\lim_{k \to -(2r+1)} \left[u(x_i, t) - \Gamma\left(\frac{k+1}{2}\right) t^{1-k} w(x_i, t) \right], \tag{20}$$

where $u(x_i, t)$ is given by (12) and $w(x_i, t)$ is found from (12) by replacing k by $2-k$ and $f(x_i)$ by $[(r+1)!2^{2r+2}]^{-1} \Delta^{r+1} f$.

Consider the number of continuous derivatives d of the initial value function $f(x_i)$. To get u from (12), take $d \geqq (m+3-k)/2$ with $k = -(2r+1)$. To solve the problem (19), take

$$\begin{aligned} d &\geqq 2(r+1) + \max\{2, (m+3-(z-k)/2)\} \\ &= \max\{3-k, (m+3-k)/2\} \qquad \text{for } k = -(2r+1). \end{aligned}$$

It was assumed earlier that M has $2r+4 = 3-k$ continuous derivatives. Thus, a sufficient condition for (20) to give a solution of (1), (2) for $k = -(2r+1)$ is that

$$d \geqq \max\{3-k, (m+3-k)/2\}.$$

To find the behavior of the solution (20) at $t=0$, note that the terms of order t^n with $n < 2r+2$ in (15) are unaffected by the subtraction process. The next term, on the other hand, gives a multiple of

$$\Delta^{r+1} f(x_i) \lim_{k \to -(2r+1)} \frac{t^{2r+2} - t^{1-k}}{2r+1+k} = \Delta^{r+1} f(x_i) t^{2r+2} \log t. \tag{21}$$

Thus, the $(2r+2)$nd t derivative of the solution (19) has a logarithmic infinity at $t=0$ unless $\Delta^{r+1} f = 0$. If $\Delta^{r+1} f = 0$, $w = 0$ and no logarithmic term appears. This is the particular case already discovered by Weinstein [26].

6. **A generalization of Huygens' principle.** Consider the special case of (12) in which k is a non-negative integer, while at the same time $m-k$ is a positive odd integer, so that $[\Gamma((k-m-1)/2)]^{-1}=0$. Then one is left with only the finite sum in (12). Furthermore, the inner sum is seen to be

$$S_n = \sum_{l=0}^{n} (-1)^l \binom{n}{l} \frac{\Gamma((m+l)/2)}{\Gamma((k+l+1)/2)} = \left[\left(\frac{d}{dy}\right)^{(m-k-1)/2} \left\{ y^{(m-2)/2}(1-y^{1/2})^n \right\}\right]_{y=1}. \tag{22}$$

This obviously vanishes for $n>(m-k-1)/2$, so that one can take $p=(m-k-1)/2$ in the sum in (12) and obtain the following solution:

$$u(x_i, t) = \frac{\Gamma((k+1)/2)}{\Gamma(m/2)} \sum_{n=0}^{(m-k-1)/2} \frac{(-t)^n}{n!} S_n \frac{\partial^n M(x_i, t; f)}{\partial t^n}. \tag{23}$$

For $k=m-1$ this reduces to the solution (3) of the equation (4).

For m odd, $k=0$, one obtains a solution of the m-dimensional wave equation

$$u(x_i, t) = \frac{\pi^{1/2}}{\Gamma(m/2)} \sum_{n=0}^{(m-1)/2} \frac{(-t)^n}{n!} S_n \frac{\partial^n M(x_i, t; f)}{\partial t^n} \tag{24}$$

which depends only on the behavior of f at the ($(m-1)$-dimensional) intersection of the retrograde characteristic half-cone (with vertex at $(x_1, \cdots, x_m, t)$) with the plane $t=0$, in agreement with Huygens' principle (more precisely, the "minor premise of Huygens'," in the terminology of Hadamard [11, pp. 75–77, 238–241]). Formula (23) provides a solution with the same property for a more general class of equations. However this property is, in general, true only when the Cauchy data are given on the singular hyperplane. This is clear from the result of Hadamard [11, p. 241] that Huygens' principle cannot hold for general data (not on the singular hyperplane) when m is even.

7. **The elliptic case.** It is easily seen that a formal change of the variable t to it changes equation (1) into

$$\Delta u + u_{tt} + \frac{k}{t} u_t = 0. \tag{25}$$

For k a positive integer this is Laplace's equation in $m+k+1$ dimensions when the solution u is axially symmetric in the last $k+1$ variables, i.e.,

$$u(x_1, \cdots, x_m, \cdots, x_{m+k+1}) \equiv u(x_1, \cdots, x_m, [x_{m+1}^2 + \cdots + x_{m+k+1}^2]^{1/2})$$

and $t = [x_{m+1}^2 + \cdots + x_{m+k+1}^2]^{1/2}$.

Suppose that $f(x_i)$ is the value for z_i real of an analytic function $f(z_i)$ of the m complex variables z_i. The solution of the initial value problem consisting of (25) together with

$$u(x_i, 0) = f(x_i), \qquad u_t(x_i, 0) = 0, \tag{26}$$

is then (12) but with t replaced by it throughout. This can be shown in exactly the same manner as (12) was shown to be a solution of the initial value problem (1), (2).

For $m=1$, $k=1$, one simply obtains the well known Laplace integral [28, Exercise 1, p. 399]

$$u(x, t) = \frac{1}{\pi}\int_0^1 [f(x + i\alpha t) + f(x - i\alpha t)](1 - \alpha^2)^{-1/2} d\alpha$$
$$= \frac{1}{\pi}\int_0^1 f(x + it\cos\theta) d\theta.$$

In the same way, for $m=1$, $k=n-2>0$, one obtains [2, Exercise 1, p. 408]

$$u(x, t) = \frac{\Gamma((n-1)/2)}{\Gamma(1/2)\Gamma((n-2)/2)}\int_0^\pi f(x + it\cos\theta)\sin^{n-3}\theta d\theta.$$

Notes added in proof (October 6, 1953): 1. It may be remarked that the formula preceding (6) could be arrived at, without the generalized method of descent, by a procedure similar to that employed by Darboux [8, pp. 66–68]. 2. Since the writing of this paper, A. Weinstein has obtained another formula for the solution of the singular Cauchy problem. This new formula is more compact than formula (5) of [26] and will appear in [27]. Furthermore, E. K. Blum has independently obtained another solution for the exceptional values of k.

BIBLIOGRAPHY

1. L. Ásgeirsson, *Über eine Mittelwerteigenschaft von Lösungen homogener linearer partiellen Differentialgleichungen 2. Ordnung mit konstanten Koeffizienten*, Math. Ann. vol. 113 (1937) pp. 321–346, especially pp. 326–327.

2. H. Bateman, *Partial differential equations*, New York, Dover, 1944.

3. F. Bureau, *Sur l'intégration de l'équation des ondes*, Académie Royale de Belgique. Bulletin de la Classe des Sciences (5) vol. 31 (1945) pp. 610–624.

4. ———, *Sur l'intégration des équations linéaires aux dérivées partielles simplement hyperboliques par la methode des singularités*, Académie Royale de Belgique. Bulletin de la Classe des Sciences (5) vol. 34, pp. 480–499.

5. ———, *Intégrales de Fourier et problème de Cauchy*, Annali di Matematica (4) vol. 32 (1951) pp. 205–233, especially pp. 206–207.

6. A. L. Cauchy, *Oeuvres complètes*, Sec. 2, vol. 6, Paris, 1887, p. 78.

7. R. Courant and D. Hilbert, *Methoden der mathematischen Physik*, vol. II, Berlin, 1937.

8. G. Darboux, *Leçons sur la théorie génerale des surfaces*, 2d ed., vol. II, p. 54 ff., Paris, 1914–15.

9. J. B. Diaz and M. H. Martin, *Riemann's method and the problem of Cauchy*, II. *The wave equation in n dimensions*, Proceedings of the American Mathematical Society vol. 3 (1952) pp. 476–483.

10. L. Euler, *Institutiones Calculi Integralis*, III, Petropoli, 1770.

11. J. Hadamard, *Le problème de Cauchy et les équations aux dérivées partielles linéaires hyperboliques*, Paris, 1932.

12. F. John, *Bestimmung einer Funktion aus ihren Integralen über gewisse Mannigfaltigkeiten*, Math. Ann. vol. 109 (1934) pp. 488–520.

13. ———, *Abhängigkeiten zwischen den Flächenintegralen einer stetiger Funktion*, Math. Ann. vol. 111 (1935) pp. 541–599.

14. M. B. Kapilevic, *On an equation of mixed elliptic-hyperbolic type*, Mat. Sbornik N.S. vol. 30 (1952) pp. 11–38.

15. E. Linés Escardó, *Resolución en forma finita del problema de Cauchy sobre una hipersuperficie cualquiera en la ecuación de ondas con cualquier número de variables, y en otras notables de tipo hiperbólico con coefficientes constantes*, Collectanea Mathematica, vol. 2, 1949, pp. 3–86.

16. M. H. Martin, *Riemann's method and the problem of Cauchy*, Bull. Amer. Math. Soc. vol. 57 (1951) pp. 238–249.

17. ———, *Mémoire sur l'intégration des équations linéaires aux derivées partielles*, Journal de l'École Polytechnique, vol. 12, no. 19, 1823.

18. S. D. Poisson, Paris Mém. de l'Acad., vol. 3, 1818.

19. M. Riesz, *L'intégrale de Riemann-Liouville et la problème de Cauchy*, Conférences de la réunion internationale des mathématiciens tenue a Paris en juillet 1937, Paris, 1939, pp. 153–170.

20. ———, *L'intégrale de Riemann-Liouville et le problème de Cauchy*, Acta Math. vol. 81 (1949) pp. 1–223.

21. O. Tedone, *Sull'integrazione dell'equazione* $\partial^2\phi/\partial t^2 - \sum_1^m \partial^2\phi/\partial x_i^2 = 0$, Annali di Matematica (3) vol. 1 (1898) pp. 1–24.

22. F. Tricomi, *Sulle equazioni lineari alle derivate parziale di 2° ordine, di tipo misto*, Atti della R. Accademia Nazionale dei Lincei, 1923, Serie Quinta, Mem. Cl. Sci. Fis. Mat. Nat., vol. 14, p. 134.

23. V. Volterra, *Sur les vibrations des corps élastiques isotropes*, Acta Math. vol. 18 (1894) pp. 161–232.

24. K. Weierstrass, *Gesammelte Werke*, Berlin and Leipzig, 1894–1927, vol. 1, p. 122.

25. A. Weinstein, *On Tricomi's equation and generalized axially symmetric potential theory*, Academie Royale de Belgique. Bulletin de la Classe des Sciences vol. 37 (1951) pp. 348–358.

26. ———, *Sur le problème de Cauchy pour l'equation de Poisson et l'equation des ondes*, C. R. Acad. Sci. Paris vol. 234 (1952) pp. 2584–2585.

27. ———, *On the wave equation and the equation of Euler-Poisson*, Proceedings of the Fifth Symposium in Applied Mathematics, 1952 (in print).

28. E. T. Whittaker and G. N. Watson, *Modern analysis*, New York, 1948.

29. S. Zaremba, Rendiconti dei Accademia dei Lincei (5) vol. 14 (1915) p. 904.

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CONSTRUCTION OF SOLUTIONS AND PROPAGATION OF ERRORS IN NONLINEAR PROBLEMS

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1. **Introduction.** The purpose of this paper is to provide theoretical tools which will be useful, in practical applications, for approximating the solutions of a certain class of nonlinear equations.

2. **Definitions.** Let $(U, +, d)$ denote a mathematical system consisting of a set U on whose elements are defined (i) a binary operation, $+$, such that U is an Abelian group with respect to this operation and (ii) a metric function d, with respect to which U is a metric space. If d is invariant under translation (i.e., for any u_1, u_2, u_3, in U, $d(u_1, u_2)=d(u_1+u_3, u_2+u_3)$), then we shall call $(U, +, d)$ a *b-space.* Under these conditions we shall, for brevity, say that *U is a b-space.* The group identity will be denoted by θ, and $d(u, \theta)$ will be denoted by $\|u\|$.

If $\mathfrak{M}$ and $\mathfrak{N}$ are sets, and K is a single-valued function defined for each $\mathfrak{m}$ of $\mathfrak{M}$ and having its values (which are denoted by $K\mathfrak{m}$ or $K(\mathfrak{m})$) in $\mathfrak{N}$ then we shall say that *K maps $\mathfrak{M}$ into $\mathfrak{N}$.*

Let U and V be b-spaces and let K map U into V. We define $m(K)$ as the infimum, and $M(K)$ as the supremum, of the quantity $\|Ku_1-Ku_2\|/\|u_1-u_2\|$ taken over all u_1, u_2 in U with $u_1\neq u_2$. If $M(K)<\infty$, then we shall say that K is *bounded*. The space V is *complete* if each Cauchy sequence in V has a limit in V. If K is bounded and if, for each bounded set $\cup_\alpha u_\alpha$ in U, the set of images, $\cup_\alpha(Ku_\alpha)$, contains a sequence which converges to an element in V, then K is said to be *completely continuous*. We shall say that *V is complete for K* if, for each Cauchy sequence $\{u_n\}$ in U, the sequence $\{Ku_n\}$ has a limit in V.

Presented to the Society, December 29, 1952; received by the editors November 20, 1952.