THE EXISTENCE OF LINE INVOLUTIONS OF ORDER GREATER THAN THREE POSSESSING A LINEAR COMPLEX OF INVARIANT LINES

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Introduction. In a recent paper [1] attention was called to a new family of line involutions in S_3 furnishing examples of involutions of all orders, m, $\geqq 4$ with complexes of invariant lines of all possible orders, i, from 2 up to the maximum, $[(m+1)/2]$. Since involutions of all orders without a complex of invariant lines are known to exist, and since examples of all possible involutions of order <4 are known, the only involutions for which existence examples remain to be supplied are those whose orders are $\geqq 4$ and whose invariant lines form a linear complex. It is the purpose of this note to define a class of involutions having these properties, thus establishing the existence of line involutions corresponding to every admissible set of characteristics (m, n, i, k) [1]. In our development we shall work exclusively on the nonsingular V_4^2 in S_5 into whose points the lines of S_3 are mapped in a 1:1 way by the well known interpretation of the Plücker coordinates of a line in S_3 as point coordinates in S_5.

1. **The definition of the involution.** On V_4^2 let there be given a point O and a plane π in general position, and let λ be the line in which the tangent hyperplane to V_4^2 at O meets π. Moreover, let C_α be a curve of order α lying on V_4^2 and meeting π in $\alpha-1$ points, β of which fall on the line λ. Obviously C_α lies in an S_4 and is rational. Finally let Γ be the set of points on V_4^2 which represents a general linear complex of lines in S_3; in other words, Γ is the intersection of V_4^2 and a general S_4.

Now if P is a general point of V_4^2 (the image of a general line in S_3), then $PO\pi$ is a 4-space which meets C_α in the $\alpha-1$ fixed points common to C_α and π, and in one additional point Q which varies with P. The point Q will of course coincide with one of the fixed intersections of C_α and π if and only if $PO\pi$ is a 4-space which contains the tangent to C_α at one of these intersections. Thus, there is a unique plane $\sigma=POQ$ which passes through P and O, meets π (since it lies in an S_4 with π), and intersects C_α in a point distinct (in general) from the $\alpha-1$ intersections of C_α and π. Now σ meets V_4^2 in a conic, γ, and this conic meets the given linear complex Γ in two points, say M and N. Finally, P (which of course lies on γ) has a unique harmonic con-

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jugate, P', with respect to M and N on the conic γ. We consider the transformation that takes P to P'; this is obviously involutory. If P is distinct from M and N, i.e., if P is not in the linear complex Γ, then P and P' are necessarily distinct, and so P cannot be invariant. On the other hand, if P is in Γ, then on γ, P coincides either with M or with N and, from the elementary properties of harmonic ranges, P' must coincide with P, i.e., P is an invariant point. Hence, the points of Γ, and only those points, are invariant. Thus, the involution has a linear complex of invariant elements, as desired, and it remains to determine the order, m, of the involution and verify that it can take on any value $\geqq 4$.

2. **The order of the involution.** To determine m it is convenient to solve first for the number, k, of points on a general line l of V_4^2 which are singular, i.e. have the property that the line PP' lies entirely on V_4^2. Then we can find m at once from the formula $m=k+2i-1$ of [1]; in fact, since $i=1$ in the present case, $m=k+1$.

To find k we observe first that the line joining a point P to its image P' will lie entirely on V_4^2 if and only if the plane σ determined by P meets V_4^2 in a conic consisting of a pair of lines. Moreover, when this is the case, one of the lines must pass through O.

Now consider a general line l on V_4^2. From the nature of the involution it is evident that the points of l are in 1:1 correspondence with the points of C_α. On l there are three and only three classes of points for which σ meets V_4^2 in a pair of lines. These arise respectively when the line of V_4^2 which passes through O in σ:

1. meets l,
2. meets C_α,
3. meets π in a point distinct from any of the β intersections of λ and C_α.

In Case 1, the singular point, L, on l is unique, being in fact the intersection of l and the S_4 which is tangent to V_4^2 at O.

In Case 2, we note that the S_4 which is tangent to V_4^2 at O meets C_α in α points, consisting of the β points $Q_j^{(1)}$ common to C_α and λ, and $\alpha-\beta$ additional points, $Q_j^{(2)}$, which are not in π. The line joining each of these points to O obviously lies entirely on V_4^2. Now π and the line joining O to any of the $\alpha-\beta$ points $Q_j^{(2)}$ determine an S_4 which meets l in a point, say $L_j^{(2)}$. Similarly, π, the line joining O to any one of the β points $Q_j^{(1)}$, and the tangent to C_α at $Q_j^{(1)}$, determine an S_4 which meets l in a point, say $L_j^{(1)}$. Any of the points $L_j^{(1)}$ and $L_j^{(2)}$ determine with O and the corresponding point $Q_j^{(1)}$ or $Q_j^{(2)}$ a plane σ which passes through O, meets C_α and π, and intersects V_4^2 in a composite conic. Moreover, the $L_j^{(1)}$ and $L_j^{(2)}$ are clearly all distinct, and different from the point L obtained in Case 1. Thus the $L_j^{(1)}$ and $L_j^{(2)}$ constitute α additional singular points on l.

Finally, in Case 3, there is a unique plane of V_4^2 passing through O and meeting π in a line, namely the plane of $O\lambda$. This plane and l determine an S_4 which meets C_α in α points, R_j, including the β intersections of C_α and λ which we have already taken into account, and hence now reject. In this S_4, the 3-space lOR_j meets λ in a point, say G_j. Moreover, since l and the plane $\sigma = O\ R_jG_j$ lie in the same 3-space, σ meets l in a point, say L_j. Since σ clearly passes through O, meets l, C_α, and π, and intersects V_4^2 in a composite conic (consisting of the lines OG_j and L_jR_j) the points L_j are also singular points on l. Further, it is clear that the L_j are all distinct and different from L and any of the points $L_j^{(1)}$ and $L_j^{(2)}$. Hence they constitute $\alpha-\beta$ additional singular points in the set of singular points on l which we are enumerating.

Therefore, on l we have altogether

$$k = 1 + \alpha + (\alpha - \beta) = 2\alpha + \beta + 1$$

singular points. Hence, the order of the involution is

$$m = k + 1 = 2\alpha + \beta + 2 \qquad (\beta \leqq \alpha - 1).$$

Thus beginning with $\alpha = 1$ and $\beta = 0$ we can obtain involutions of all orders $\geqq 4$ possessing linear complexes of invariant elements.

Reference

1. C. R. Wylie, Jr., *A new series of line involutions in S_3*, Mathematics Magazine, January–February (1950) pp. 125–131.

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