ON THE GROUP OF AFFINE TRANSFORMATIONS OF AN AFFINELY CONNECTED MANIFOLD[1]

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It has been proved by S. Myers and N. Steenrod that the group of all isometries of a Riemannian manifold is a Lie group.[2] The purpose of the present note is to establish a similar theorem for the group of all affine transformations of an affinely connected manifold under the assumption of *"completeness"* which will be explained below. We do not know whether this assumption is really necessary, and with a hope of eliminating it in the future, we shall give a series of lemmas in the general case. The formulation and proof of the main theorem (Theorem 1) will be given in the last section.

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1. **Preliminaries.** Let M be a differentiable manifold[3] of dimension n with an affine connection. According to Chern's formulation, the affine connection is given by a set of $n+n^2$ linear differential forms θ^i and θ^i_j $(i, j=1, 2, \cdots, n)$ on the bundle of frames B^* of M which satisfy the equations of structure of affine connection, in the following fashion. To a local coordinate system (u^i) in M there corresponds a system of local coordinates (u^i, X^k_i) in B^* defined by the condition that the n vectors of any frame are given by $e_i = \sum_k X^k_i \partial/\partial u^k$ $(i=1, 2, \cdots, n)$, where $\det \|X^k_i\| \neq 0$. Let $\|Y^k_i\|$ be the inverse matrix of $\|X^k_i\|$. Set

$$\theta^i = \sum_j Y^i_j du^j \quad \text{and} \quad \theta^k_i = \sum_j Y^k_j\Big(dX^j_i + \sum_{l,m} \Gamma^j_{ml} X^l_i du^m\Big),$$

where Γ^j_{ml} are the so-called coefficients of the affine connection with respect to the local coordinates (u^i). Then θ^i and θ_i^k are linearly independent linear differential forms defined on the whole space B^* and satisfy the following equations of structure:

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[2] S. Myers and N. Steenrod, *The group of isometries of a Riemannian manifold*, Ann. of Math. vol. 40 (1939).

[3] Differentiability means that of class C^∞, though our results hold for weaker assumptions. On the other hand, we do not assume the second axiom of countability for M.

$$d\theta^j - \sum_k \theta^k \wedge \theta^j_k = \frac{1}{2}\sum_{l,m} P^j_{lm}\theta^l \wedge \theta^m,$$

$$d\theta^j_i - \sum_l \theta^l_i \wedge \theta^j_l = \frac{1}{2}\sum_{l,m} S^j_{ilm}\theta^l \wedge \theta^m,$$

where

$$\frac{1}{2} P^j_{lm} = \sum_{i,p,q} Y^j_i X^p_l X^q_m T^i_{pq},$$

$$S^j_{ilm} = \sum_{k,p,q,r} Y^j_k X^p_i X^q_l X^r_m R^k_{pqr},$$

T^i_{pq} and R^k_{pqr} being the components of the torsion and curvature tensors respectively.

If ϕ is a differentiable homeomorphism of M onto itself, it induces a differentiable homeomorphism of the tangent bundle B and the bundle of frames B^* onto themselves, which we shall denote by $D\phi$. $D\phi$ leaves the forms θ^i invariant. We shall call ϕ an *affine transformation* if $D\phi$ leaves the forms θ^i_j invariant.

For any element $b=(p, X)$ of B, where $p\in M$ and X is an element of the tangent vector space T_p of p, there exists a path (auto-parallel curve) $C_b(t)$, defined in $-\epsilon<t<\epsilon$ for some $\epsilon>0$, with origin p and tangent vector at p which is equal to X. With respect to a local coordinate system (u^i) with origin p, $C_b(t)$ is expressed by a set of solutions of the system of equations

$$(*) \qquad d^2f^i/dt^2 + \sum_{j,k} \Gamma^i_{jk} df^j/dt \cdot df^k/dt = 0,$$

corresponding to the initial conditions

$$f^i(0) = 0, \qquad (df^i/dt)_0 = \alpha^i,$$

where $X = \sum \alpha^i \partial/\partial u^i$. A parameter t satisfying these conditions is uniquely determined and will be called the canonical parameter of the path C_b. We denote by Ω^a the set of points $b\in B$ such that the corresponding path $C_b(t)$ can be extended over the values of the canonical parameter $-\epsilon<t<a+\epsilon$ for some $\epsilon>0$. For each $b=(p, X)$ $\in\Omega^1$ we denote the point $C_b(1)$ by $\eta(b)=\eta(p, X)$ or $\eta_p(X)$. For each $p\in M$, let Ω^a_p be the set of elements $X\in T_p$ such that $b=(p, X)\in\Omega^a$.

LEMMA 1. *For any $s>0$ and $a>0$, we have*

$$s\Omega^a_p = \Omega^{a/s}_p \qquad (p\in M).$$

LEMMA 2. *For each* $a>0$, Ω^a *is an open subset of* B. *For each* $p\in M$, Ω_p^a *is an open neighborhood of* 0 *in* T_p.

LEMMA 3. $b\to\eta(b)$ *is a differentiable mapping of* Ω^1 *into* M.

These lemmas, whose setup is due to Chevalley, follow from the properties of solutions of (*). By a fundamental existence theorem in the theory of differential equations, there exist $\epsilon_0>0$ and $\delta_0>0$ such that the solutions $f^i(x, \alpha, t)$ of (*) with the initial conditions

$$f^i(x, \alpha, 0) = x^i, \qquad (df^i(x, \alpha, t)/dt)_{t=0} = \alpha^i,$$

exist uniquely for $-\epsilon_0<t<\epsilon_0$, provided that $|x^i|<\delta_0$ and $|\alpha^i|<\delta_0$. Since we have $f^i(x, \alpha, st)=f^i(x, s\alpha, t)$ for small $s>0$, we see that the solutions $f^i(x, \alpha, t)$ are defined for $-\epsilon<t<1+\epsilon$ for some $\epsilon>0$, provided that $|x^i|<\delta_0$ and $|\alpha^i|<\delta_0\epsilon_0$. We set $F^i(x, \alpha)=f^i(x, \alpha, 1)$ for $|x^i|<\delta_0$ and $|\alpha^i|<\delta_0\epsilon_0$. $F^i(x, \alpha)$ are differentiable functions in x and α. We also remark the following. Suppose that the path $f^i(x, \alpha, t)$ exists in a coordinate neighborhood for $-\epsilon<t<a+\epsilon$ for some $a>0$. Then for any x and α which are sufficiently near x_0 and α_0 respectively, a path $f^i(x, \alpha, t)$ exists for $-\epsilon<t<a+\epsilon$ and $f^i(x, \alpha, t)$ and $(df^i(x, \alpha, t)/dt)_{t=a}$ are continuous in x and α. We shall now prove the above lemmas.

PROOF OF LEMMA 1. Let $X\in\Omega_p^a$ and let $C_b(t)$, $-\epsilon<t<a+\epsilon$, be the corresponding path. We define a path $C'(t)$ by $C'(t)=C_b(st)$ for $-\epsilon/s<t<a/s+\epsilon/s$. $C'(t)$ has the tangent vector at p which is equal to sX and therefore $sX\in\Omega_p^{a/s}$. This being true for any $s>0$, we see that $s\Omega_p^a=\Omega_p^{a/s}$.

PROOF OF LEMMA 2. It is sufficient to prove it for $a=1$. Let $b=(p_0, X_0)\in\Omega^1$ and let $C_b(t)$ be the corresponding path which is defined for $-\epsilon<t<1+\epsilon$ for some $\epsilon>0$. We cover the path by a finite number of coordinate neighborhoods $N(p_0), N(p_1), \cdots, N(p_k)$ such that $p_1=C_b(t_1)$, $p_2=C_b(t_1+t_2)$, $\cdots$, $p_k=C_b(1)$ and $p_{i+1}\in N(p_i)$ for each $i=0, 1, \cdots, k-1$. Let $\{x^1, x^2, \cdots, x^n\}$ be a coordinate system in $N(p_0)$, and let $p_0=(0, 0, \cdots, 0)$ and $X_0=(\alpha_0^1, \alpha_0^2, \cdots, \alpha_0^n)$. The path $C_b(t)$, $-\epsilon'<t<t_1+\epsilon'$ for small $\epsilon'>0$, lies in $N(p_0)$. By the remark we have made above, there exist $\delta_1, \delta_2>0$ such that for $q=(x^1, x^2, \cdots, x^n)$ and $Y=(\alpha^1, \alpha^2, \cdots, \alpha^n)$ with $|x^i|<\delta_1$ and $|\alpha_0^i-\alpha^i|<\delta_2$, a path with origin q and tangent vector Y at q exists for $-\epsilon'<t<t_1+\epsilon'$. Furthermore, the point and the tangent vector of this path corresponding to $t=t_1$ are sufficiently near those of the path $C_b(t)$, $-\epsilon'<t<t_1+\epsilon'$, if q and Y are sufficiently near p_0 and X_0 respectively. Using this argument backward starting from $N(p_{k-1})$, we see that if q and Y are sufficiently near p_0 and X_0 respectively, the

path corresponding to (q, Y) can be defined for $-\epsilon<t<1+\epsilon$. This proves that the set Ω^1 is open. The second statement is obvious.

Lemma 3 can be easily proved from what we have done so far.

Let $F^i(x, \alpha)$ be as before. It is easy to show that $(\partial F^i(x, \alpha)/\partial\alpha^j)_{\alpha=0} = \delta^i_j$, that is, the differentiable mapping $(x, \alpha)\rightarrow(x, F(x, \alpha))$ has non-zero Jacobian at (0, 0). Hence there exists a neighborhood W of 0 such that the above mapping is a differentiable homeomorphism of $W\times W$ onto its image V which is an open set containing (x, x), $x\in W$ (remark that $F(x, 0)=x$). We shall denote by $(x, y)\rightarrow(x, G(x, y))$ the inverse mapping which is also differentiable. We take a neighborhood N of 0 such that $N\times N\subset V$. Then, for any $x, y\in N$ there exists $\alpha\in W$ such that $y=F(x, \alpha)$. We can take N as small as we wish.

Now let p_0 be any fixed point of M. By taking a coordinate system (u^i) with origin at p_0, we can consider N as a neighborhood of p_0. For any two points $p=(x^i)$ and $q=(y^i)$ in N, there exists $\alpha\in W$ such that $F(x, \alpha)=y$, which means that there exists a tangent vector X at p such that $\eta(p, X)=q$. When p is fixed, $q\rightarrow\eta_p^{-1}(q)$ is a differentiable homeomorphism of N onto a neighborhood of 0 in T_p, which we shall denote by N_p. We call such a neighborhood N a *regular neighborhood* in M. We have thus shown that any point of M is contained in an arbitrarily small regular neighborhood.

Let N be a regular neighborhood and p, q two points in N. For any $X\in N_p$, $\eta(p, X)$ is a point in N and hence is representable uniquely as $\eta(q, Y)$ for some $Y\in N_q$. We denote the mapping $X\in N_p\rightarrow Y\in N_q$ so defined by π_{pq}. It is a differentiable homeomorphism of N_p onto N_q with the inverse mapping π_{qp}.

LEMMA 4. *Let N be a regular neighborhood and let K be a compact set contained in N. Then the set $K^*=\bigcup_{p,q\in K}(p, \eta_p^{-1}(q))$ is a compact set in B.*

PROOF. This is clear since the mapping $(p, q)\in N\times N\rightarrow(p, \eta_p^{-1}(q))\in B$ is continuous.

We have previously shown that the set Ω^1 is an open set in B. We shall say that an affinely connected manifold M is *complete* if Ω^1 coincides with the whole B. If this is the case, the set Ω^a coincides with B for every $a>0$ as we see from Lemma 1. We see therefore that M is complete if and only if every path $C_b(t)$, $b\in B$, can be extended for any large value of the canonical parameter. A complete Riemannian manifold is complete in our sense when considered as an affinely connected manifold.

2. **Affine transformations.** An affine transformation of M clearly

maps any path into a path. From this fact we have

LEMMA 5. *Let N be a regular neighborhood in M. Let p_0 be any fixed point in N. Any affine transformation ϕ can be expressed in N by*

$$\phi(p) = \eta(\phi(p_0), D_{p_0}\phi(\overset{-1}{\eta}_{p_0}(p))), \qquad p \in N,$$

where $D_{p_0}\phi$ denotes the mapping of T_{p_0} into $T_{\phi(p_0)}$ which is induced by ϕ.

LEMMA 6. *Let ϕ be an affine transformation of M such that $\phi(p_0)=p_0$ for a certain point p_0. If $D_{p_0}\phi$ is the identity mapping of the tangent space T_{p_0}, then ϕ is the identity transformation on M. In particular, if an affine transformation leaves a nonempty open set of M pointwise fixed, it is the identity transformation.*

PROOF. If $\phi(p_0)=p_0$ and if $D_{p_0}\phi$ is the identity transformation of T_{p_0}, then ϕ is the identity mapping on a regular neighborhood N of p_0 as we see from Lemma 5. For any point q of M, we can take a finite number of regular neighborhoods $N(p_i)$ such that $P_{i+1}\in N(p_i)$ for each $i=0, 1, \cdots, k-1$ and $p_k=q$. Then we see that ϕ is the identity transformation in each $N(p_i)$ and hence $\phi(q)=q$, which proves that ϕ is the identity transformation on M.

We shall say that a sequence of differentiable mappings ϕ_n of M into itself *converges strongly* to a differentiable mapping ϕ of M into itself if ϕ_n converges to ϕ over M, $D\phi_n$ to $D\phi$ over B, and $DD\phi_n$ to $DD\phi$ over the tangent bundle of B. Then the following is an almost immediate consequence of the definition of affine transformation.

LEMMA 7. *If a sequence of affine transformations of M converges strongly to a differentiable homeomorphism ϕ of M onto itself, then ϕ is an affine transformation.*

We shall now prove

LEMMA 8. *Assume that M is complete. Let ϕ_n be a sequence of affine transformations which satisfy the following conditions at a certain point p_0 of M:*

(1) *$\phi_n(p_0)$ converge to a certain point p';*

(2) *$D_{p_0}\phi_n$ converge to a certain linear mapping Λ of T_{p_0} into $T_{p'}$.*[4]

Then there exists a differentiable mapping of M into itself which is the strong limit of ϕ_n on M.

PROOF. Let N be a regular neighborhood of p_0. We shall first prove that there exists a differentiable mapping of N into M which is the strong limit of ϕ_n on N. We define $\phi(p)=\eta(p', \Lambda(\eta_{p_0}^{-1}(p)))$ for $p\in N$.

[4] More precisely, the mappings $D_{p_0}\phi_n$ of T_{p_0} into B: $X\in T_{p_0}\to(\phi_n(p_0), (D_{p_0}\phi_n)(X))$ converge to the mapping $X\in T_{p_0}\to(p', \Lambda X)\in B$.

The right-hand side is well defined since M is complete. Since $\eta_{p_0}^{-1}$ and $\eta_{p'}$ are differentiable mappings and since Λ is a linear mapping, ϕ is a differentiable mapping of N into M. It is also clear from definition that $D_{p_0}\phi=\Lambda$. Since ϕ_n are affine transformations, we have by Lemma 5

$$\phi_n(p) = \eta(\phi_n(p), (D_{p_0}\phi_n)(\eta_{p_0}^{-1}(p))) \qquad \text{for } p \in N.$$

We see thus that $\phi_n(p)$ converge to $\phi(p)$ for every $p\in N$. As for $D\phi_n$ and $D\phi$ we have

$$D\phi_n = (D\eta_{\phi_n(p)})(D_{p_n}\phi_n)(D\eta_{p_0}^{-1})$$

and

$$D\phi = (D\eta_{p'})(D_{p_0}\phi)(D\eta_{p_0}^{-1}).$$

It is easy to show that $D\eta_{\phi_n(p)}$ converge to $D\eta_{p'}$. Therefore we see that $D\phi_n$ converge to $D\phi$ over N. Similarly, we can prove that $DD\phi_n$ converge to $DD\phi$ over N.[5]

In order to extend ϕ to the whole M, consider the totality of pairs (U, ϕ_U), where U is an open set containing N and ϕ_U is a differentiable mapping of U into M which is the strong limit of ϕ_n over U. By defining the partial order of extension as usual and using Zorn's lemma, we see that there exists a maximal (U, ϕ_U). If U is not equal to M, we can find a regular neighborhood N which has a point, say p_1, in common with U and which does not entirely lie in U. Then the assumptions of our lemma being satisfied at this point p_1, we see that ϕ_U can be extended to a differentiable mapping of $U\cup N$ into M which is the strong limit of ϕ_n, which contradicts the maximality of (U, ϕ_U). This concludes the proof.

The following lemma can be easily proved.

LEMMA 9. *Let ϕ_n and ψ_n be sequences of affine transformations. If ϕ_n converge strongly to a differentiable mapping ϕ on M, then for any point p and any neighborhood V' of $\phi(p)$, there exists a neighborhood V of p such that $\phi_n(V)\subset V'$ for sufficiently large n. If ϕ_n and ψ_n converge strongly to differentiable mappings ϕ and ψ respectively on M, then $\phi_n\psi_n$ converge to $\phi\psi$ on M.*

3. **Theorem.** We shall now formulate the main theorem. Let M be an affinely connected manifold. Let $A(M)$ be the group of all affine transformations of M. If we introduce the compact-open topology

[5] In these lines care must be taken of the meaning of the mappings and their differential mappings involved, as in the footnote 4.

into $A(M)$, it is a topological transformation group consisting of differentiable homeomorphisms of M.

THEOREM 1. *If M is complete, $A(M)$ is a Lie group.*

In virtue of a theorem of S. Bochner and D. Montgomery,[6] our theorem will be established if we prove that $A(M)$ is locally compact and that any element of $A(M)$ which leaves a nonempty open subset of M pointwise fixed is the identity element. The last condition has been proved in Lemma 6. We shall prove that $A(M)$ is locally compact.

Let N be a regular neighborhood of any arbitrary but fixed point p_0 of M. Take two neighborhoods U and V of p such that $\overline{U}\subset V$, $\overline{V}\subset N$, and $\overline{V}$ is compact and contained in a coordinate neighborhood. We shall prove that the closure $\overline{W}$ of the neighborhood $W=\{\phi \in A(M) | \phi(\overline{U})\subset V, \phi^{-1}(\overline{U})\subset V\}$ of the identity element of $A(M)$ is compact. Before this, we introduce a metric topology into $\overline{W}$ as follows: if $\phi, \psi\in\overline{W}$ are expressed by a set of functions $f^i(x)$ and $g^i(x)$ respectively with respect to any fixed coordinate system covering V, we set

$$\rho(\phi, \psi) = \max_{1\leqq i,j\leqq n} \{ | f^i(p_0) - g^i(p_0) |, \ | \partial f^i/\partial x_j - \partial g^i/\partial x_j |_{p_0} \}$$

and $\rho^*(\phi, \psi)=\rho(\phi, \psi)+\rho(\phi^{-1}, \psi^{-1})$. Then ρ and ρ^* are metrics in $\overline{W}$ in virtue of Lemma 6. It is not difficult to show that the ρ^*-topology in $\overline{W}$ is stronger than the topology in $\overline{W}$ which is induced by the compact-open topology of $A(M)$. Thus it is sufficient to prove that $\overline{W}$ is compact in the ρ^*-metric.

Let $\phi_n\in\overline{W}$. Since $\overline{V}$ is compact, the set $V^*=\bigcup_{p,q\in\overline{V}}(p, \eta_p^{-1}(q))$ in the tangent bundle B is compact by Lemma 4. Since $\phi_n(\overline{U})\subset V$, we see that $D_{p_0}\phi_n$ are mappings of $\overline{U}_{p_0}(=\eta_{p_0}^{-1}(\overline{U}))$ into V^*. V^* being compact, we can take a subsequence such that $D_{p_0}\phi_{n_\nu}$ converge to a certain mapping which maps (p_0, X) $(X\in T_{p_0})$ into $(p', \Lambda(X))\in V^*$, where p' is a certain point in $\overline{V}$ and Λ is a linear mapping of T_{p_0} into $T_{p'}$. We then know by Lemma 8 that there exists a differentiable mapping ϕ of M into itself which is the strong limit of ϕ_{n_ν}. Since $\phi_{n_\nu}^{-1}(\overline{U})\subset V$, we can take a subsequence again and assume that there exists a differentiable mapping ψ of M into itself which is the strong limit of $\phi_{n_\nu}^{-1}$. ϕ and ψ being the strong limit of ϕ_{n_ν} and $\phi_{n_\nu}^{-1}$ respectively, we see by Lemma 9 that $\phi\psi$ and $\psi\phi$ are the identity transformation, that is, ϕ and ψ are differentiable homeomorphisms which are inverse

[6] S. Bochner and D. Montgomery, *Locally compact groups of differentiable transformations*, Ann. of Math. vol. 47 (1946).

to each other. By Lemma 7, ϕ is an affine transformation of M. Clearly $\phi \in \overline{W}$. We also see that ϕ is the limit of ϕ_n in the ρ^*-metric in $\overline{W}$. We have thereby proved that $\overline{W}$ is compact in the ρ^*-metric and hence in the compact-open topology of $A(M)$. This completes the proof of our theorem.

The following is evident.

THEOREM 2. *Let M be a Riemannian manifold and consider the affine connection which is induced by the metric. Then the group of all isometries $I(M)$ is a closed subgroup of $A(M)$.*

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