ON A CLASS OF OPERATORS[1]

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1. **Introduction.** The purpose of this note is to establish a structure theorem (Theorem 1) for operators on a Hilbert space satisfying the relation $AA^*A = A^*A^2$ or (using the notation $A \leftrightarrow B$ to indicate that A and B are commutative operators) $A \leftrightarrow A^*A$. Such an operator is resolved in a simple way into constituents which are either normal or isometric and we shall say accordingly that it possesses property i.n. The terminology and notation are standard and will be found to conform, for the most part, for instance, with [1].[2] Some points that perhaps deserve particular mention, however, are: Hilbert space means Hilbert space over the complex numbers; an operator is a bounded linear transformation of a Hilbert space into itself; the symbol 1 denotes the identity operator; and a subspace is always closed. If M is a linear submanifold which is not necessarily closed we use the notation M^- for the subspace which it generates (its topological closure). A *suboperator* of an operator T is a direct summand of T, i.e., the result of restricting T to a reducing subspace.

2. **Isometries.**[3] Let H be a Hilbert space and V an isometry on H. If L is a subspace of H so is $V(L)$. In particular, $V(H)$ is a subspace. The orthogonal complement $H \ominus V(H)$ we call the *deficiency* of V and denote M_0. If L_0 is a subspace of M_0, then we may define the subspaces L_n, $n \geqq 0$, of H by induction, setting $L_{n+1} = V(L_n)$. The L_n's are pair-wise orthogonal. We call their span $\sum_n \oplus L_n$ the *slice* L through L_0 (with respect to V). Slices reduce V.

The orthogonal complement of the special slice M through the deficiency M_0 we denote N. M and N reduce V. If $M = 0$, V is unitary. If $N = 0$ we may call V *pure*. In general, $V = V_1 \oplus V_2$ where $V_1 = V|M$, $V_2 = V|N$. V_1 and V_2 are the *purely isometric* and *unitary parts* of V respectively.

If K is any Hilbert space we use the notation $\tilde{K}$ for the Hilbert

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[2] Numbers in brackets refer to the list of references at the end of the paper.

[3] This § (including Lemma 2.1) is only a recapitulation of the known theory of linear isometries. A thoroughgoing exposition including proofs may be found in [3].

space of all sequences $\{x_n\}_{n=0}^{\infty}$ of elements of K with $\sum_{n=0}^{\infty}\|x_n\|^2<\infty$ and with the usual Hilbert space structure. Also, the operator S_K on $\tilde{K}$ which carries the sequence $\{x_0, x_1, \cdots\}$ into $\{0, x_0, x_1, \cdots\}$ we call the *shift operator defined by* K. Set $\phi_n = V^n | M_0$ (writing $V^0=1$). Then ϕ_n: $M_0 \to M_n$ is an isomorphism for $n \geqq 0$ and the map $\phi = \sum_n \oplus \phi_n$ of $\tilde{M}_0$ onto M is also an isomorphism. We may now state the preparatory

LEMMA 2.1. *Let V be an isometry on H and let M_0 be its deficiency. Then V is the direct sum of two suboperators V_1 and V_2 defined on the subspaces M and N respectively* (*where $M \oplus N = H$ and M is the slice through M_0*) *such that V_1 is purely isometric and V_2 is unitary. This decomposition is unique. The purely isometric part is unitarily equivalent under ϕ to the shift operator S_{M_0}.*

3. Commuting projections.

LEMMA 3.1. *Let V be an isometry on H and let $V = V_1 \oplus V_2$ be its resolution into purely isometric and unitary parts. Then a reducing subspace for V is a direct sum $L_1 \oplus L_2$ where L_1 is a slice with respect to V_1 and L_2 is a subspace of N which reduces V_2. Equivalently, if E is a projection commuting with V then $E = E_1 \oplus E_2$ where E_1 is a projection of M onto a slice and E_2 is a projection on N which commutes with V_2.*

PROOF. It is easy to verify that VV^* is a projection and that M_0 is its null space. It follows that $1 - VV^*$ is the projection of H onto M_0, so this projection is in the ring[4] $\boldsymbol{R}$ generated by V. Replacing V by V^{k+1} in this remark and noting that the deficiency of V^{k+1} is $M_0 \oplus M_1 \oplus \cdots \oplus M_k$, we see that $\boldsymbol{R}$ contains the projections of H onto $M_0 \oplus \cdots \oplus M_k$ for $k \geqq 0$ and hence also the projections onto M and N.

Now let E be a projection such that $E \leftrightarrow V$. Then $E \leftrightarrow \boldsymbol{R}$ and M and N reduce E. Write $E_1 = E|M$, $E_2 = E|N$ so that $E = E_1 \oplus E_2$. Then $E_1 \leftrightarrow V_1$, $E_2 \leftrightarrow V_2$ and the proof is reduced to showing:

LEMMA 3.2. *The reducing subspaces of a pure isometry V are exactly the slices.*

PROOF. By Zorn's Lemma it suffices to show that every nontrivial reducing subspace contains a nontrivial slice. Accordingly, let $L \neq 0$ reduce V. Since L clearly contains the slice through $L \cap M_0$ it will be enough to prove that $L \cap M_0 \neq 0$. But now the projection onto M_0, $1 - VV^*$, is also reduced by L so that the assumption $L \cap M_0 = 0$ leads to the conclusion $L \perp M_0$. That this is contradictory is readily

[4] I.e., a $*$-algebra of operators closed in the sense of the weak operator topology.

seen by writing an arbitrary vector $x \in L$ as $x = \sum_{k=0}^{\infty} x_k$ with $x_k \in M_k$. The condition then implies $x_0 = 0$. But $V^{*k}x$ is also in L and its component in M_0 is $V^{*k}x_k = 0$ so that x_k is in the deficiency of V^k as well as in M_k. Since we have seen this deficiency to be $M_0 \oplus \cdots \oplus M_{k-1} \perp M_k$ it follows that $x_k = 0$ for all k so that $x = 0$ and $L = 0$ contrary to assumption.

4. **Property i.n.** We shall say that an operator A possesses *property i.n.* if $A \leftrightarrow A^*A$. While the choice of this characterization is clearly dictated by its immediacy and brevity, our working definition is formulated below. It has seemed well to state this result in a more general form than the purpose of this note requires.

LEMMA 4.1. *The following are mutually equivalent conditions on an arbitrary operator A:*

(1) *A has property i.n.*
(2) *In the polar resolution $A = WP$, $W \leftrightarrow P$.*
(3) *$A = VP = PV$ with $P \geqq 0$ and V isometric.*
(4) *$A = VB = BV$ with B self-adjoint and V isometric.*

PROOF. (1) implies (2): Let $A = WP$ be the polar resolution of A. Then $P = (A^*A)^{1/2}$ and it follows that P commutes with A. This can be written $PWP = WP^2$ or $(PW - WP)P = 0$ which says that $PW - WP$ annihilates the subspace $P(H)^-$, whence it is clearly sufficient to show that it also annihilates $H \ominus P(H)$. But since P is self-adjoint this latter subspace is the null space of P, as is well known, and what we must show is merely $PWy = WPy$ when $Py = 0$, which is clear from the definition of the polar resolution.

(2) implies (3): By definition W is a partial isometry and the null space of P is $H \ominus S$ where S is the initial space of W, whence $S = P(H)^-$. But this implies that S reduces W and the desired isometry V can be taken to be $V = (W|S) \oplus W'$ where W' is any isometry defined on $H \ominus S$.

(3) implies (4): Obvious.

(4) implies (1): $A^*A = B^2$.

Now let A be an operator on H with property i.n. Then also $A^* \leftrightarrow A^*A$ so that if $A^*Ax = 0$ we must have

$$A^*A^*Ax = 0 = A^*A(A^*x)$$

and the null space Z of A^*A, which is also the null space of A, reduces A. Write $A = A_1 \oplus A_2$ where $A_1 = A|H \ominus Z$ and $A_2 = A|Z = 0$. The operator A_1 itself possesses property i.n. Accordingly, for the time being, let us assume $A = A_1$, $Z = 0$, and let $A = VP$ be the polar resolution of A. Then V is an isometry and (Lemma 4.1) $V \leftrightarrow P$ so that the subspaces M, N, and M_n for $n \geqq 0$ all reduce P. In particular $P = P_1 \oplus P_2$ where as usual $P_1 = P | M$ and $P_2 = P | N$ and P_1 and P_2 commute with V_1 and V_2 respectively. The operator $V_2 P_2$ on N is normal and we turn attention to the other part $V_1 P_1$.

As is appropriate we shall simplify notation once more, assuming temporarily that $V_1 = V$ and $P_1 = P$, so that the isometry V is pure. Now P is reduced by each M_n. Write $P | M_n = P_n$ and $P = \sum_n \oplus P_n$. The fact that P commutes with V can then be exactly expressed by saying that each P_n is carried into the operator P_0 under the isomorphism $\phi_n: M_0 \to M_n$. The isomorphism ϕ between $\tilde{M}_0$ and H implements the unitary equivalence of P itself and the operator $\tilde{P}_0 = P_0 \oplus P_0 \oplus \cdots$ (defined by $\tilde{P}_0\{x_n\} = \{P_0 x_n\}$) and hence makes $A = VP$ correspond to the mapping

$$\{x_0, x_1, \cdots\} \to \{0, P_0 x_0, P_0 x_1, \cdots\}.$$

Let us introduce some terminology. If T is any operator on a Hilbert space K, then on the space $\tilde{K}$ we have the shift operator S_K and the operator $\tilde{T}$ defined by T as above. Their product $\hat{T} = S_K \tilde{T} = \tilde{T} S_K$ will be called the *dilated shift operator* defined by T. This notion permits us to state briefly that VP is unitarily equivalent under ϕ to the dilated shift operator $\hat{P}_0$.

Putting these parts together again we see that if A has property i.n., then $A = B \oplus C \oplus 0$ where C is normal and B is unitarily equivalent to a dilated shift operator. Letting $C' = C \oplus 0$ we may say more succinctly: $A = B \oplus C'$ with C' normal. Note that the null spaces of A and C' are the same. This fact along with the fact that the dilated shift operator unitarily equivalent to B is defined by a positive operator will now be shown to characterize the given decomposition.

LEMMA 4.2. *The dilated shift $\hat{T}$ defined by T is normal if and only if T (or, what comes to the same thing, $\hat{T}$) is* 0.

PROOF. Apply both $\hat{T}^*\hat{T}$ and $\hat{T}\hat{T}^*$ to a vector $\{x, 0, 0, \cdots\}$ in the deficiency of S_K. One obtains (assuming normality) $T^*Tx = 0$ whence $Tx = 0$. Since x is arbitrary in K we conclude $T = 0$.

Now suppose $\hat{P}$ is the dilated shift operator defined by a positive operator P and suppose $\hat{P}$ (or, what comes to the same thing, P) has no null space. Then the expression $S_K \tilde{P}$ defining $\hat{P}$ is its polar decomposition and the pure isometry S_K is in the ring generated by $\hat{P}$. This means that if E is a projection commuting with $\hat{P}$, then $E \leftrightarrow S_K$ and hence a reducing subspace of $\hat{P}$ is a slice with respect to S_K. If we identify K with the subspace of $\tilde{K}$ consisting of all sequences $\{x, 0, 0, \cdots\}$, then this slice is the slice through some subspace L_0 of K and the suboperator of $\hat{P}$ determined by the reducing subspace is readily seen to be the dilated shift defined by the suboperator $P|L_0$.

Putting these last remarks together we see that a dilated shift operator which is defined by a positive operator and which has no null space can have no nontrivial normal suboperators, and hence, regarding the decomposition $A = B \oplus C'$ given above, we are able to say that C' is the normal kernel of A, i.e., its largest normal suboperator. This shows that the decomposition is unique. Moreover, because of the uniqueness of the polar resolution it is seen that in the expression of B as the dilated shift operator defined by the positive operator P_0, the P_0 employed is also unique up to unitary equivalence.

Finally, it is recalled that P_0 was obtained by writing $A_1 = A|H \ominus Z$ and setting $P_0 = P|M_0$ where VP is the polar resolution of A_1 and M_0 is the deficiency of V. The operator P is then $(A_1^*A_1)^{1/2}$ and it is not hard to see that $P_0 = (A^*A)^{1/2}|M_0$ while

$$M_0 = (H \ominus Z) \ominus V(H \ominus Z)$$

and $V(H \ominus Z) = A(H \ominus Z)^- = A(H)^-$ so that

$$M_0 = (H \ominus Z) \ominus A(H)^- = H \ominus (Z \oplus A(H)^-).$$

We have proved

THEOREM 1. *Let the operator A on the Hilbert space H possess property i.n. and let Z be its null space. Then A can be written as the direct sum $B \oplus C$ of a normal operator C and an operator B which is unitarily equivalent to the dilated shift operator defined by a positive operator P_0. It is possible so to choose this decomposition that the null space Z belongs entirely to C. Under this additional assumption the decomposition is unique, C being the normal kernel of A and P_0, which is unique up to unitary equivalence, can be taken to be the positive square root of the suboperator $A^*A|H \ominus (Z \oplus A(H)^-)$.*

COROLLARY. *A complete set of unitary invariants for an operator with property i.n. is provided by the unitary equivalence class of the suboperator $A^*A|H \ominus (Z \oplus A(H)^-)$ along with that of its normal kernel.*

PROOF. Clear from the theorem.

REMARK. This serves to make equally clear of course the nature of an operator A which commutes with AA^*, or in other words the adjoint of an operator with property i.n. Needless to say these two conditions are not at all the same. Indeed, the only way both A and its adjoint can possess property i.n. is for A to be normal.

In conclusion we may add that various mild side conditions on an operator with property i.n. will insure that it be normal. For instance, it is not difficult to see that an operator which generates a finite ring (in the sense of Murray and von Neumann [2]) and has property i.n. is normal, as is any completely continuous operator with property i.n. Both these remarks apply in particular of course to operators on finite-dimensional Hilbert spaces.

References

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