LATTICE PACKING IN THE PLANE WITHOUT CROSSING ARCS

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Introduction. We first review some definitions and results of Chalk and Rogers.[2] If S and T are two sets of points in Euclidean n-space, then $S+T$ will denote the set of all points $s+t$ where s is in S and t is in T, while $S-T$ is composed of all $s-t$. The point set sum will be denoted by $S\cup T$ and the intersection by $S\cap T$. Let Λ be a lattice; then $S+\Lambda$ is a lattice packing if no two sets $S+\lambda$ and $S+\lambda'$, with λ and λ' distinct points in Λ, have a common point in their interiors. Let $D(S)$ be the set of all $s-s'$ where s and s' are points in the interior of S. Chalk and Rogers have shown that $S+\Lambda$ is a lattice packing if and only if the lattice Λ is admissible for $D(S)$, i.e., has no point in the interior of $D(S)$ except possibly the origin O. If S is convex this criterion reduces to a result of Minkowski,[3] well known especially when S is symmetric. For, if S is also open, then $D(S)=S-S=S+S=2S$, this being the set S expanded by a factor 2.

From now on we assume that all point sets lie in the plane. We wish to provide a similar criterion for the situation in which no arc of $S+\lambda$ crosses an arc of $S+\lambda'$. Then $S+\Lambda$ is called a lattice packing without crossing arcs, and is in particular a lattice packing.

Since $D(S)$ omits from consideration all arcs of S not in the interior of S, it is to be expected that it will be of no use for our purpose. Our criterion will refer instead to the set $E(S)$ defined as $S-S$. Clearly $E(S)$ is symmetric. It is easy to see that $E(S)$ is generated by translating S such that always one of its points is at O. If $E(S)$ is a circle and its interior then S is a figure of constant breadth and conversely.

If S is the square $0\leqq x\leqq 1$, $0\leqq y\leqq 1$, and T consists of three sides of this square, omitting the side on the y-axis, then $E(S)=E(T)$. Yet there are lattices which provide lattice packings without crossing arcs for T but which do not do so for S; e.g., the lattice generated by $(1/2, 0)$ and $(0, 1)$. The condition we obtain will involve local properties of $E(S)$ arising from S.

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[2] J. H. H. Chalk and C. A. Rogers, *The critical determinant of a convex cylinder*, J. London Math. Soc. vol. 23 (1948) pp. 178–187; see pp. 186–187.

[3] H. Minkowski, *Dichteste gitterformige Lagerung kongruenter Körper*, Nachr. Akad. Wiss. Göttingen. Math.-Phys. Kl., 1904, pp. 311–355; see p. 313.

We first discuss the crossing of arcs in general and obtain criteria applicable to our problem.

1. **Crossing arcs.** Denote by $e(A)$ the point set consisting of the two end points of an arc A.

DEFINITION. Let A, B be arcs with a common subarc M. Suppose that for every neighborhood $V\supset M$ there are arcs A', $B'\subset V$ and an arc D satisfying the following conditions.

(i) $A\supset A'\supset M$; $B\supset B'\supset M$; $e(D) = d(B')$;

(ii) $D\cap(A'\cup B') = e(B')$;

(iii) A meets the interior of the closed curve $B'\cup D$ in a point a_1 and the exterior at a point a_2.

Then we say that A *V-crosses* B. If also

(iv) $$e(B')\cap A' = \emptyset$$

then we say that A *crosses* B; and if in addition

(v) $$e(A')\cap B' = \emptyset,$$

we say that A *s-crosses* B.

If M consists of a single point, we speak of a *point-crossing*, *point-V-crossing*, or *point-s-crossing*.

Of these only the s-crossing is symmetric. An example of A crossing B but B not crossing A occurs when A is the x-axis from -1 to 2, M is the x-axis from 0 to 1, and B is M and $y=x\sin 1/x$ $(-1\leqq x<0)$.

LEMMA 1. *If A s-crosses B at M, then B s-crosses A at M.*

PROOF. Let A', B', a_1, a_2, D be as in the definition. Join the end points of A' by an arc D_2 which meets A' nowhere else. Then $D_2\cup A'$ is a simply closed curve. Now $B'\cap D_2\cap A'=\emptyset$. Hence there is a subarc B_1 of B such that $M\subset B_1$, $B_1\cap D_2=\emptyset$, and $e(B_1)\cap A'=\emptyset$. Such a subarc B_1 may be obtained by taking any subarc in the interior of the largest subarc B_3 of B containing M whose interior is disjoint from D_2, but itself containing in its interior the largest subarc of B_3 with end points in A'.

If B_1 has points both interior and exterior to $D_2\cup A'$, then B s-crosses A at M. Otherwise suppose that no point of B_1 is interior to $D_2\cup A'$.

Choose a neighborhood of M sufficiently small that its boundary meets A' in points separated by M and likewise for B'. In W there are arcs A_1', B_1', D_1 satisfying the definition of s-crossing. Necessarily $A_1'\subset A'$, $B_1'\subset B'$; let a_1' be a point of A_1' lying interior to $B_1'\cup D_1$, and a_2' a point of A_1' exterior. Let ϵ be the smaller of the distance of A_1' from D_1 and of a_1', a_2' from $B_1 \cup D_1$; thus $\epsilon > 0$. Choose points p_1, p_2 interior to $D_2 \cup A'$ but with the distance of p_1 to a_1' and p_2 to a_2' less than ϵ. Thus p_1 is interior to $B_1' \cup D_1$ and p_2 exterior. Join p_1 to p_2 by an arc E lying interior to $D_2 \cup A'$ and every point of which is less than distance ϵ from A_1'. Then E does not intersect D_1, but has points interior and exterior to $D_1 \cup B_1'$. Hence it intersects B_1' and a fortiori B'. But since E lies interior to $D_2 \cup A'$ it cannot intersect B'. This contradiction shows that the assumption that B_1 had no points interior to $D_2 \cup A'$ is false.

A very similar argument shows that we cannot assume that B_1 has no point exterior to $D_2 \cup A'$. Thus B_1 has points both interior and exterior to $D_2 \cup A'$. Hence B_1 s-crosses A at M.

LEMMA 2. *If A does not cross B at M, then A V-crosses B at M if and only if A crosses B in every neighborhood of M.*

The proof for "if" is immediate.

Conversely, suppose A V-crosses, but does not cross, B at M. Then for some neighborhood V of M the points a_1, a_2 as given in the definition of V-crossing will be the end points of a subarc A_1 of A which does not contain M. In going from a_1 to a_2 on A_1 let m be the last point which lies on B such that no preceding point lies exterior to $B' \cup D$, and let M_1 be the component of $A' \cap B'$ containing m. This component M_1 is contained in the interior of B' since $e(B') \cap A = \emptyset$. Also M_1 is contained in the interior of A_1 since it lies between a_1 and a_2 and does not contain them.

We show finally that A crosses B at M_1. Let V_1 be a neighborhood of M_1. By our construction the component of $B \cap V_1$ which contains M_1 has points b_1, b_2 separated by M_1 on B and not in A. Let B_1' be the subarc of B with end points b_1, b_2. Let D_1 be the rest of the simple closed curve $B \cup D$; then $B_1 \cup D_1$ and $B \cup D$ are the same simple closed curves. Let r_1 be the distance of M_1 from D_1; hence $r_1 > 0$. Let V be the set of points whose distance from M_1 is less than r_1, and let A_1' be the component of M_1 in $A \cap \overline{V}$ ($\overline{V}$ denotes the closure of V). Then by the construction A' contains points a_1', a_2' separated by M_1 and not in B, where a_2' is exterior to $B \cup D$ and a_1' is of necessity interior. Then in the definition of A crossing B at M_1 use V_1, A_1', B', a_1', a_2'.

THEOREM 1. *If an arc A crosses an arc B, then $B - A$ contains a neighborhood of the origin.*

Let A', B', D, a_1, and a_2 be as described in the definition. Let A''

be the subarc of A' joining a_1 and a_2. Let r be the smallest of the following distances: of a_1 from $B'\cup D$, of a_2 from $B'\cup D$, and of A'' from D. Translate A'' by an amount $r'<r$ in any direction θ to obtain an arc A_1; then A_1 cannot meet D. Now a_1 goes to a point a_1' and a_2 to a_2'; also a_1' is interior to $B'\cup D$ while a_2' is exterior. Hence A_1 intersects $B'\cup D$ and must do so in B' and at a point b which is the image under the translation of a point a of A. The point $b-a$ has polar coordinates (r', θ). Thus $B-A$ contains the interior of the circle of radius r and with center at the origin.

The converse of the theorem is not true. As a counterexample let A and B both be the spiral given in polar coordinates by $r=1/\theta$ $(1\leqq\theta\leqq\infty)$ and M be the origin. Another counterexample is given by letting A be the double spiral $r=1/\theta$ and $r=1/(\theta+\pi/2)$ and B the double spiral $r=1/(\theta+\pi)$ and $r=1/(\theta+3\pi/2)$, where throughout $1\leqq\theta\leqq\infty$.

Yet another counterexample occurs when A and B are the same triod S; i.e., S consists of three arcs, having an end point p in common but no other point common to any two of them.

LEMMA 3. *If S is a triod then $E(S)$ contains a neighborhood of the origin.*

Suppose that S consists of the three arcs A, B, C, each having p as an end point and no two having any other point in common. Let the other end points be a on A, b on B, and c on C. Let D be a simple closed curve containing a, b, c but no other points of A, B, C, and with p inside D. Let D_{ab} be the subarc of D, with end points a and b and not containing c; D_{bc} and D_{ca} are similarly defined. Then $D=D_{ab}\cup D_{bc}\cup D_{ca}$. Also $D_{ab}\cup A\cup B$ is a simple closed curve.

The remainder of the proof is similar to that of Theorem 1. Let a' be a point on the interior of A and let A' be the subarc of A joining p and a'. Define b', B' and c', C' similarly. Let r_a be the lesser of the distances of a' from $B\cup C$ and A' from D, with corresponding definitions for r_b and r_c. Take r as the minimum of r_a, r_b, r_c. Then $r>0$.

Translate $F=A'\cup B'\cup C'$ by an amount $r'<r$ in any direction θ, obtaining the set F_1. Then p goes into a point p_1 which must lie inside D. But p_1 must be inside or on the boundary of one of the simple closed curves $D_{ab}\cup A\cup B$, $D_{bc}\cup B\cup C$, or $D_{ca}\cup C\cup A$; assume it is so for the first curve. Then the image c_1' of c' lies outside that simple closed curve and the image C_1' of C', having one end point outside and other inside or on the boundary of $D_{ab}\cup A\cup B$, must intersect it at a point w which is also the image z_1 of a point z on C'. This intersection w cannot be on D; hence it lies on $A\cup B$. Thus $w-z=z_1-z$

is the point with polar coordinates (r', θ) and lies in $E(S)$. This shows that $E(S)$ contains the circle of radius r and center at 0, as was to be proved.

THEOREM 2. *Let B be an arc for which $E(B)$ has no point with vectorial angle θ except the origin, i.e., no secant of B has inclination θ. Suppose M is a subarc in the interior of B and also of an arc A. Then A does not cross B at M if and only if there are a neighborhood V containing M and a direction α such that no point in $(B\cap V)-(A\cap V)$ has vectorial angle α except the origin.*

The "if" part is a consequence of the previous theorem.

If A does not cross B at M there is a subarc A' of A and a subarc B' of B, both containing M but with end points different from those of M, and an arc D joining the end points of B' and not otherwise intersecting A' or B' such that no point of A' lies in the interior of the simple closed curve $B'\cup D$. Choose a subarc B'' of B' containing M but with end points distinct from those of B'. Let d be the distance of B'' from D. Let W be the set of points with polar coordinates (d', θ), $0<d'<d/2$. None of the points of $B''+W$ lies on D since $d'<d/2$ and none lies on B by the hypothesis on θ. The same is true of $B''-W$. Thus either $B+W_1$ or $B-W_1$ lies in the interior of $B'\cup D$; let it be denoted by $B+W$. Let $\alpha=\theta+\pi$ or θ according as $W=W_1$ or $-W_1$. Then $B''-W$ lies interior to $B'\cup D$ and hence cannot intersect A'. Thus $B''-A'$ does not contain W. Finally, if V is taken such that its distance from M is $<d/2$, we see that the theorem is satisfied.

2. **Local boundary point.** Before returning to the problem of lattice packings, we introduce another term.

DEFINITION. A point p is called a *local boundary point* of $A+B$ if (1) p lies in the closure Cl $(A+B)$ of $A+B$, and (2) whenever $p=a+b$ with a in A and b in B, there exist neighborhoods U of a and V of b such that p is a boundary point of $(A\cap U)+(B\cap V)$.

By a local boundary point of $E(S)$ is meant a local boundary point of $S+(-S)$.

If S is composed of three sides of a square, then $E(S)$ is a square and the locus of its local boundary points consists of the four sides and also the two lines joining the midpoints of opposite sides.

LEMMA 4. *If A is open, then $A+B$ contains none of its local boundary points.*

For if $p=a+b$ with a in A and b in B, then for every neighborhood U of a with $U\subset A$, $U+b$ is a neighborhood of p and $U+b\subset A+B$.

Hence for any neighborhood V of b, $(U\cap A)+(V\cap B)=U+(V\cap B)$ contains the neighborhood $U+b$ of p since b is in $V\cap B$. Thus p cannot be a local boundary point.

The converse, that if $A+B$ contains none of its local boundary points then either A or B is open, is not true. Counterexample: $A=B$ and is the punctured disc consisting of every point whose distance d from the origin satisfies $1\leqq d<4$.

3. Lattice packings.

THEOREM 3. *In order that $S+\Lambda$ be a lattice packing without point-crossing arcs it is sufficient that the points of Λ distinct from* 0 *and belonging to $E(S)$ be local boundary points of $E(S)$.*

The proof is by contradiction. Suppose that there are arcs A and B in S such that $A+\lambda'$ crosses $B+\lambda''$, where λ' and λ'' are distinct points of Λ. Then A and $B+\lambda$ cross, where $\lambda=\lambda''-\lambda'$. Hence by Theorem 1, $A-(B+\lambda)$ contains a neighborhood of 0 and A and $B+\lambda$ may be be taken to lie in any neighborhood, no matter how small, of the point of crossing. Thus $A-B$ contains a neighborhood of λ. Also $A-B$ lies in $E(S)$. Hence λ is not a local boundary point of $E(S)$. But λ is a point of Λ distinct from 0. The existence of such a point is a contradiction to the hypothesis on the nonexistence of such points, because of Lemma 2.

If in Theorem 3 crossings on common subarcs, not simply points, are to be excluded, then the following modification needs to be made. Whenever two subarcs A, B of S can be made to coincide by a translation, i.e., $A=B+p$ for some point p, then in determining whether p or $-p$ is a local boundary point only those neighborhoods of A or B are used which include maximal arcs A_1 or B_1 such that $A_1\subset A$, $B_1\subset B$, and $A_1=B_1+p$.

Although Theorem 2 provides a partial converse to Theorem 1, yet a partial converse of Theorem 3 cannot be obtained from Theorem 2 without imposing rather stringent hypotheses.

An example illustrating some of the difficulty is the following. On the interval $-\epsilon\leqq x\leqq\epsilon$, let S be the set of all horizontal line segments with y rational and between $-\epsilon$ and ϵ and let T be the same except that y is irrational or 0. Then $S-T$ contains a neighborhood of the origin and the origin is not a local boundary point, although S and T do not have crossing arcs.

Theorem 2 could be used if one required that on every arc A of S every point has a neighborhood in which the secants of A do not have all directions and by modifying the definition of a local boundary point so that instead of $U \cap A$ and $V \cap B$ one takes any subarc of A in U containing a and any subarc of B in V containing b. These changes are felt to be undesirable in being too great a departure from the original concept of $D(S)$ and in excluding sets with nonempty interiors.

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THE DEGREE FORMULA FOR THE SKEW-REPRESENTATIONS OF THE SYMMETRIC GROUP[1]

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1. **Introduction.** In his paper on the representations of the symmetric group,[2] G. de B. Robinson defines certain "skew-representations" and associates these to skew-diagrams (to be defined below) analogously to the way the irreducible representations of the symmetric group are associated with regular diagrams. Furthermore he shows that the degree of such a skew-representation is equal to the number of orderings of the related skew-diagram.[3]

The object of this note is to derive a formula for the degree of skew-representation related to a given skew-diagram.[4] This problem will be treated strictly in terms of the number of orderings of such a diagram, and from this point of view is very similar to the question attacked in [5] by R. M. Thrall.

In §4, this formula is applied to the problem of computing the characters of certain classes of the symmetric group.

2. **Definitions and lemmas.** A partially ordered set P is said to be *regular* or a *regular diagram* if:

(I) The elements of P may be represented by ordered pairs of integers (i, j), $i>0, j>0$, where $(i, j) \leqq (p, m)$ if and only if $i \leqq p$ and $j \leqq m$, $(i, j)=(p, m)$ if and only if $i=p$ and $j=m$,

(II) $\max_i (i, j) \leqq \max_i (i, j')$ whenever $j \geqq j'$,

(III) $(i, k) \in P$ implies $(j, k) \in P$ for all integers j with $1 \leqq j \leqq i$,

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[1] The work on this paper was performed under the sponsorship of the O.N.R.

[2] See [1; 2; 3; 4].

[3] See [1, p. 290].

[4] This is an answer to the question raised in [1, p. 294], the ϕ of that paper is the g of the theorem below.